Chemical Engineering Science 65 (2010) 3431–3438

Contents lists available at ScienceDirect

Chemical Engineering Science

journal homepage: www.elsevier.com/locate/ces

A design method for robust and quadratic optimal MIMO linear controllers

M.A. Garcia-Alvarado [a,*], I.I. Ruiz-López [b]

[a] Chemical and Biochemical Engineering Department, Instituto Tecnológico de Veracruz, Av. Miguel Angel de Quevedo 2779, Veracruz, Ver. 91860, Mexico

[b] Faculty of Chemical Engineering, Benemérita Universidad Autónoma de Puebla, Av. San Claudio y 18 Sur S/N, Puebla, Pue. 72570, Mexico

ARTICLE INFO

Article history:
Received 8 October 2008
Received in revised form
23 November 2009
Accepted 17 February 2010
Available online 25 February 2010

Keywords:
Optimal control
Robust control
Dynamic simulation
MIMO linear controllers
CSTR control

ABSTRACT

In this paper a design method was formulated to deal with robustness and performance specifications for any MIMO linear controller. The controller tuning procedure was expressed as an optimization problem in which novel time-domain integrals of the weighted squared error and weighted squared control signals, with initial state zero and inputs not necessarily defined over the Lebesgue normed space ($\mathcal{L}_{2+}$), were minimized. The control robustness is achieved by constraining the minimization such that the maximum complex/real ratio of the closed-loop control system eigenvalues was lower than one. The proposed tuning method was applied in the design of linear controllers with PID structure for a CSTR with disturbance noise and a nonlinear CSTR with control signal saturations, both reported in literature. The results show that the proposed control systems surpass the performance and robustness characteristics of the controllers designed with other reported methods.

1. Introduction

PID algorithm is a linear control action that has been one of the most successful algorithms in the control theory history. Even with the introduction of non-linear and robust control theory, the PID control action has demonstrated good performance and robustness characteristics for non-linear processes (Tan et al., 2002a, b; García-Alvarado et al., 2005; Cheng and Chiu, 2008). Even more, some non-linear controllers have PID configuration, like the proposed by Alvarez et al. (1989) which structure corresponds to a PI controller with gain and integral time as functions of state variables. Furthermore, the robustness properties of PI and PID algorithms have been demonstrated (Bao et al., 1999; Alvarez et al., 1998; Chen et al., 2002; Ge et al., 2002; Toscano, 2005; Ruiz-López et al., 2006; Xiong et al., 2007; Goncalves et al., 2008). The PID control action is a particular case of a general linear controller, and therefore a higher order linear controller must keep and may improve its performance and robustness characteristics.

A process with a MIMO linear controller may be represented by,

$$\frac{dx}{dt} = Ax + B_1 w + B_2 u \tag{1}$$

$$y = C_1 x + D_{11} w + D_{12} u \tag{2}$$

$$\frac{d\xi}{dt} = \mathcal{A}\xi + \mathcal{B}_1 r + \mathcal{B}_2 y \tag{3}$$

$$u = \mathcal{C}\xi + \mathcal{D}_1 r + \mathcal{D}_2 y \tag{4}$$

Where $x \in \boldsymbol{R}^{n\times 1}$ is the process state vector, $w \in \boldsymbol{R}^{m\times 1}$ is the exogenous input vector (disturbance signals), $y \in \boldsymbol{R}^{r\times 1}$ is the measured output vector (feedback to the controller), $u \in \boldsymbol{R}^{c\times 1}$ is the control signal vector, $\xi \in \boldsymbol{R}^{k\times 1}$ is the control state vector, and $r \in \boldsymbol{R}^{r\times 1}$ is the set point vector. Eqs. (1) and (2) represent the state-space of the process and Eqs. (3) and (4) represent the state-space of the control algorithm, in which is implicit the error signal vector $e \in \boldsymbol{R}^{r\times 1} = r - y$. If A, B_i, C_i and D_{ij} are constants, the process is linear-invariant, otherwise the system may be linear time-dependent, quasi-linear or non-linear. If $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{C}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ are constants the controller is linear. A PID control action with first order filter can be written in the form of Eqs. (3) and (4) as will be shown later.

A common method for designing an optimal controller is by evaluation of the control signal (u) that minimize a quadratic performance index with the form,

$$I = \int_0^\infty [x'Qx + u'Ru]\, dt \tag{5}$$

or

$$I = \int_0^\infty [y'Qy + u'Ru]\, dt \tag{6}$$

In the case where the process is linear, the minimization of Eq. (5) or (6) can be achieved by applying the Riccati equation. In this way, the results may be a general linear controller (Engwerda and Weeren, 2008), or the parameters for a PI algorithm (García-Alvarado et al., 2005). If the process is non-linear, the problem may be solved by other techniques, as the Chebyshev

* Corresponding author.
E-mail address: miguelg@itver.edu.mx (M.A. Garcia-Alvarado).

0009-2509/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ces.2010.02.033

spectral approximation, and the results are generally non-linear controllers (El-Kady et al., 2003). However the minimization of a quadratic performance index does not assure controller robustness.

Nowadays, one of the most common design methods for an optimal and robust control of the system represented by Eqs. (1)–(4) is based on solving optimization problems in the Hardy normed spaces $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$. Doyle et al. (1989) showed that a unique solution for matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ exists that minimize the $\mathcal{H}_2$-norm of the closed-loop matrix function of Eqs. (1)–(4). Moreover, Doyle et al. (1989) deduced suboptimal solutions that assure $\mathcal{H}_2$-norm $<\gamma_1$ and $\mathcal{H}_\infty$-norm $<\gamma_2$, where $\mathcal{H}_\infty$ is the infinite norm of closed-loop matrix function of Eqs. (1)–(4) and γ_1 and γ_2 are any real positive number. Each one of the suboptimal problems, is solved from the solutions of two Riccati equations. Because $\mathcal{H}_2$-norm represents a quadratic performance index, and $\mathcal{H}_\infty$-norm represents a robustness index, the suboptimal problems can be used in order to equilibrate $\mathcal{H}_2/\mathcal{H}_\infty$ norms (that is, the tradeoff between performance and robustness). Other solutions in normed spaces $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$ may be obtained from the application of linear matrix inequalities (LMI) (Chilali et al., 1999). As example, Saeki (2006) proposed a direct application of LMI for the MIMO PID controller design as an optimization problem in which matrix inequalities obtained from a $\mathcal{H}_\infty$-norm condition for each frequency are approximated by LMIs, and then a sequence of PID gains is obtained iteratively until reaching a $\mathcal{H}_\infty$-norm lower than one.

$\mathcal{H}_2/\mathcal{H}_\infty$ methods have been applied in the design of chemical engineering process controllers (Bao et al., 1999; Chen et al., 2002; Ge et al., 2002; Toscano, 2005; Goncalves et al., 2008). Bao et al. (1999) proposed an application of LMIs for the design of a MIMO PID controller applied to a distillation column. This study showed the dependence of transient response of the control system designed with LMIs on weighting function selection (Saeki, 2006). Ge et al. (2002), Toscano (2005) and Goncalves et al. (2008), studied the control of a continuous stirred tank reactor (CSTR), which dynamic was described by Uppal et al. (1974). Ge et al. (2002) applied a classical PID algorithm and founded the control parameters in such way that the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are kept under a desired value using LMIs. Toscano (2005) also applied a classical PID algorithm and proposed a simple way to evaluate the control parameters with robustness characteristics. Toscano's method consisted in finding the control parameters that minimize the sensitivity function $\|\mathbb{S}(s)\|_\infty$ norm (similarly to Bao et al., 1999) and keep a pseudo-damping factor in a lower bound. Goncalves et al. (2008) introduced a noise disturbance in the same CSTR and applied an ISA (Instrument Society of America) PID configuration with noise filter. The ISA PID configuration is basically a PI with a weighted derivative action. They found the set control parameters that minimize a weigh function of $\|\mathbb{T}(s)\|_2$ and $\|\mathbb{T}(s)\|_\infty$. Goncalves et al. (2008) reported explicitly that their method increase the performance and robustness of the CSTR control with respect to Ge et al. (2002) and Toscano (2005) methods. In Section 4, the Goncalves et al. (2008) control will be described in detail and represented in the general linear control form Eqs. (3) and (4). Chen et al. (2002) designed a MIMO PI control for a CSTR with two target output variables, two control signals, and two exogenous inputs via LMIs. Although the control tuned by Chen et al. (2002) exhibited a good performance, Ruiz-López et al. (2006) founded another ser of controls parameters for the same system with better characteristics for both reference tracking and disturbance rejection. The method applied by Ruiz-López et al. (2006) represents an alternative to LMIs that requires lower numerical effort and did not depend on weighting functions. They showed that the minimization of the eigenvalues maximal complex/real ratio (Im/Re) for the closed-loop dynamic characteristic matrix is equivalent to the minimization of the $\mathcal{H}_\infty$-norm and therefore imparts robustness properties to the control systems. In order to assure the controller performance, Ruiz-López et al. (2006) minimized iteratively the quadratic index in Eq. (6) with $R=0$, $r=0$, $w=0$ and $x(0)=x_0\neq 0$ which are the most common conditions used for the minimization of quadratic performance indexes (Engwerda and Weeren, 2008; García-Alvarado et al., 2005; El-Kady et al., 2003). However, an important characteristic of the system defined by Eqs. (1)–(3) is that if, $r=0$, $w=0$ and $x(0)=x_0\neq 0$, then it reduces to an homogeneous state-space, so if system is internally stable, the state-space trajectories tends to zero when time approaches infinite even without a control action. Therefore, the time-domain integral for error and control signals when the system is subjected to input disturbances and $x(0)=0$, may represent better the control performance. That is, the following quadratic index,

$$I_e=\int_0^\tau [(r-y)'Q(r-y)+u'Ru]\,dt \tag{7}$$

for Eqs. (1)–(4) and with

$$r\neq 0\notin L_{2+},\quad w\neq 0\notin L_{2+},\quad \text{and}\quad x(0)=0 \tag{8}$$

The set point (r) and exogenous input (w) were declared non-elements of Lebesgue normed space ($\mathcal{L}_{2+}$) in order to consider the inclusion of a step input as forcing function. The analytical solution of integral (7) under conditions (8) is not reported, and therefore in this paper such analytical solution was deduced for a general linear controller and applied in the design of linear controllers. This design method was expressed as a minimization problem of the deduced solution for integral (7) subjected to a constraint in the maximum Im/Re ratio of the closed-loop control system eigenvalues, as robustness index. The method was validated by tuning the CSTR SISO control described by Goncalves et al. (2008) and the CSTR MIMO control described by Chen et al. (2002) and later improved by Ruiz-López et al. (2006).

2. Theory

The process defined by Eqs. (1) and (2) with control declared in Eqs. (3) and (4) can be rewritten as the following closed-loop equations,

$$X'=[x'\quad \xi'] \tag{9}$$

$$\frac{dX}{dt}=\mathbb{A}X+\mathbb{B}_1 w+\mathbb{B}_2 r \tag{10}$$

$$y=\mathbb{C}_1 X+\mathbb{D}_{11} w+\mathbb{D}_{12} r \tag{11}$$

$$u=\mathbb{C}_2 X+\mathbb{D}_{21} w+\mathbb{D}_{22} r \tag{12}$$

where

$$\mathbb{A}=\begin{bmatrix} A+B_2\mathcal{D}_2\Delta_1 C_1 & B_2\Delta_2\mathcal{C} \\ \mathcal{B}_2\Delta_1 C_1 & \mathcal{A}+\mathcal{B}_2\Delta_1 D_{12}\mathcal{C}\end{bmatrix},$$

$$\mathbb{B}_1=\begin{bmatrix} B_1+B_2\mathcal{D}_2\Delta_1 D_{11} \\ \mathcal{B}_2\Delta_1 D_{11}\end{bmatrix},\quad \mathbb{B}_2=\begin{bmatrix} B_2\Delta_2\mathcal{D}_1 \\ \mathcal{B}_1+\mathcal{B}_2\Delta_1 D_{12}\mathcal{D}_1\end{bmatrix}$$

$$\mathbb{C}_1=[\Delta_1 C_1\quad \Delta_1 D_{12}\mathcal{C}],\quad \mathbb{C}_2=[\mathcal{D}_2\Delta_1 C_1\quad \Delta_2\mathcal{C}]$$

$$\mathbb{D}_{11}=[\Delta_1 D_{11}],\quad \mathbb{D}_{12}=[\Delta_1 D_{12}\mathcal{D}_1],\quad \mathbb{D}_{21}=[\mathcal{D}_2\Delta_1 D_{11}],\quad \mathbb{D}_{22}=[\Delta_2\mathcal{D}_1]$$

$$\Delta_1=(I_r-D_{12}\mathcal{D}_2)^{-1},\quad \Delta_2=(I_c+\mathcal{D}_2\Delta_1 D_{12})$$

Therefore, the closed-loop transfer matrix (from w and r inputs to y as output) can be expressed as,

$$\mathbb{T}_{wr,y}(s) = \mathbb{C}(I_{n+k}s - \mathbb{A})^{-1}\mathbb{B} + \mathbb{D} \text{ with } \mathbb{C} = [\mathbb{C}_1], \quad \mathbb{B} = [\mathbb{B}_1 \ \ \mathbb{B}_2], \quad \mathbb{D} = [\mathbb{D}_{11} \ \ \mathbb{D}_{12}] \tag{13}$$

Ruiz-López et al. (2006) showed that the minimization of the maximal Im/Re relation (called $\phi_{\max}$) of eigenvalues of matrix $\mathbb{A}$ is equivalent to the minimization of the $\mathcal{H}_\infty$-norm of matrix transfer of Eq. (16) ($\|\mathbb{T}_{wr,y}(s)\|_\infty$) and therefore assures the robustness properties of the control systems. Ruiz-López et al. (2006) recommended a $\phi_{\max}$ value lower than one.

The analytical solution of problem (10)–(12) subjected to certain given inputs and $X(0)=0$ is required to solve integral (7). A forcing function that can approximate a feasible perturbation in the input or reference signals is a step function, which expressed in Laplace dominion are given by,

$$w(s) = s^{-1}K_1, \quad r(s) = s^{-1}K_2 \tag{14}$$

where $K_1 \in \boldsymbol{R}^{m\times 1}$ and $K_2 \in \boldsymbol{R}^{r\times 1}$ contain the magnitude of the step functions. Under Eq. (17) the input signals are not defined in a Lebesgue space ($w(t), r(t) \notin \mathcal{L}_{2+}$), and therefore if the matrix transfer of the process does not contain a zero-pole, a controller with integral action (or a zero pole) is required to guarantee that $e(t) \to 0$ when $t \to \infty$ (or $e(t) \in \mathcal{L}_{2+}$) as in the PID algorithm. The solution of Eqs. (10)–(12) subject to inputs (14) is,

$$X = \mathbb{A}^{-1}[e^{\mathbb{A}t} - I_{n+k}][\mathbb{B}_1K_1 + \mathbb{B}_2K_2] \tag{15}$$

And therefore,

$$y = \mathbb{C}_1\mathbb{A}^{-1}[e^{\mathbb{A}t} - I_{n+k}][\mathbb{B}_1K_1 + \mathbb{B}_2K_2] + \mathbb{D}_{11}K_1 + \mathbb{D}_{12}K_2 \tag{16}$$

$$u = \mathbb{C}_2\mathbb{A}^{-1}[e^{\mathbb{A}t} - I_{n+k}][\mathbb{B}_1K_1 + \mathbb{B}_2K_2] + \mathbb{D}_{21}K_1 + \mathbb{D}_{22}K_2 \tag{17}$$

Quadratic index (10) can be split in two integrals,

$$I_e = \int_0^\tau [(r-y)'Q(r-y) + u'Ru]\,dt = I_y + I_u \tag{18}$$

Each one of the integrals in Eq. (18) may be evaluated for the servomechanism or regulator problems. In a servomechanism problem no exogenous inputs are assumed ($K_1=0$), and therefore,

$$I_{y,servo} = \int_0^\tau (K_2 - y)^T Q (K_2 - y)\,dt \tag{19}$$

In a regulator problem no reference changes are assumed ($K_2=0$), and therefore,

$$I_{y,reg} = \int_0^\tau y^T Q y\,dt \tag{20}$$

The analytical evaluation of integrals (19) and (20) is summarized in the following theorems.

Theorem 1. *For system defined with Eqs.* (10)–(12) *in minimal realization, with a zero pole in* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, *and with* $\mathbb{A}$ *stable* (*the whole of its eigenvalues must be in left complex semi plane*), *the integral* (19) *is given by*,

$$\lim_{\tau\to\infty} I_{y,servo} = K_2'\mathbb{B}_2'\mathbb{P}_y\mathbb{B}_2K_2 \tag{21}$$

where $\mathbb{P}_y$ *is obtained by solving the Riccati equation*,

$$\mathbb{A}'\mathbb{P}_y + \mathbb{P}_y\mathbb{A} = -(\mathbb{A}^{-1})'\mathbb{C}_1'Q\mathbb{C}_1\mathbb{A}^{-1} \tag{22}$$

Proof. See Appendix A.

Theorem 2. *For system defined by Eqs.* (10)–(12) *in minimal realization, with a zero pole in* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, *and with* $\mathbb{A}$ *stable, the integral* (20) *is given by*,

$$\lim_{\tau\to\infty} I_{y,reg} = K_1'\mathbb{B}_1'\mathbb{P}_y\mathbb{B}_1K_1 \tag{23}$$

where $\mathbb{P}_y$ *is obtained solving the Riccati Eq.* (22)

Proof. See Appendix B.

In the analytical evaluation of I_u is necessary to consider that $u(t)$ may not be in $\mathcal{L}_{2+}$ under condition given by Eq. (14), and therefore is possible that $I_u \to \infty$ when $\tau \to \infty$. Thus, I_u must be evaluated using a finite value for τ. Then, the analytical evaluation for the performance index of control variables (I_u) is obtained by applying the results of Theorems 1 and 2. The results are,

$$\begin{aligned} I_{u,servo} = \int_0^\tau u'Ru\,dt &= K_2'\mathbb{B}_2'\mathbb{P}_u\mathbb{B}_2K_2 - K_2'\mathbb{B}_2'(e^{\mathbb{A}\tau})'\mathbb{P}_u e^{\mathbb{A}\tau}\mathbb{B}_2K_2 \\ &+ 2(\mathbb{A}^{-1})(\mathbb{D}_{21}K_2 - \mathbb{C}_2\mathbb{A}^{-1}\mathbb{B}_2K_2)'R\mathbb{C}_2\mathbb{A}^{-1}(e^{\mathbb{A}\tau} - I_{n+k})\mathbb{B}_2K_2 \\ &+ (\mathbb{D}_{21}K_2 - \mathbb{C}_2\mathbb{A}^{-1}\mathbb{B}_2K_2)'R(\mathbb{D}_{21}K_2 - \mathbb{C}_2\mathbb{A}^{-1}\mathbb{B}_2K_2)\tau \end{aligned} \tag{24}$$

$$\begin{aligned} I_{u,reg} = \int_0^\tau u'Ru\,dt &= K_1'\mathbb{B}_1'\mathbb{P}_u\mathbb{B}_1K_1 - K_1'\mathbb{B}_1'(e^{\mathbb{A}\tau})'\mathbb{P}_u e^{\mathbb{A}\tau}\mathbb{B}_1K_1 \\ &+ 2(\mathbb{A}^{-1})(\mathbb{D}_{21}K_1 - \mathbb{C}_2\mathbb{A}^{-1}\mathbb{B}_1K_1)'R\mathbb{C}_2\mathbb{A}^{-1}(e^{\mathbb{A}\tau} - I_{n+k})\mathbb{B}_1K_1 \\ &+ (\mathbb{D}_{21}K_1 - \mathbb{C}_2\mathbb{A}^{-1}\mathbb{B}_1K_1)'R(\mathbb{D}_{21}K_1 - \mathbb{C}_2\mathbb{A}^{-1}\mathbb{B}_1K_1)\tau \end{aligned} \tag{25}$$

where $\mathbb{P}_u$ is obtained solving the Riccati equation

$$\mathbb{A}'\mathbb{P}_u + \mathbb{P}_u\mathbb{A} = -(\mathbb{A}^{-1})'\mathbb{C}_2'R\mathbb{C}_2\mathbb{A}^{-1} \tag{26}$$

Another important performance indicator in a control system is the capacity to reject noise. This capacity may be simulated with the response to a unit-impulse (Dirac delta $\delta(t)$) in the exogenous input,

$$w(s) = \mathbf{1}, \quad r(s) = \mathbf{0} \text{ where } \mathbf{1}' \in R^{1\times m} = [1 \ \ 1 \ \ \cdots \ \ 1]$$

Analytical solution of Eq. (10) under these inputs is,

$$X = e^{\mathbb{A}t}\mathbb{B}_1\mathbf{1} \tag{27}$$

And therefore,

$$y = \mathbb{C}_1 e^{\mathbb{A}t}\mathbb{B}_1\mathbf{1} + \mathbb{D}_{11}\boldsymbol{\delta}(t) \tag{28}$$

$$u = \mathbb{C}_2 e^{\mathbb{A}t}\mathbb{B}_1\mathbf{1} + \mathbb{D}_{21}\boldsymbol{\delta}(t) \tag{29}$$

where: $\boldsymbol{\delta}(t)' \in R^{1\times m} = [\delta(t) \ \ \delta(t) \ \ \cdots \ \ \delta(t)]$

Under Eqs. (28) and (29) the quadratic performance indexes are finites only if $D_{11}=0$, due by the properties of Dirac delta function,

$$\int_{-\infty}^{\infty} \delta(t)\delta(t)\,dt = \delta(0) \to \infty$$

Then, the quadratic performance indexes only can be evaluated assuming $D_{11}=0$. This assumption is similar to those required by Doyle et al. (1989) for solving the sub-optimal problems for $\mathcal{H}_2/\mathcal{H}_\infty$ norms. However in the proposed performance evaluation, this assumption does not represent a loss of generality because in the case of $D_{11} \neq 0$, its effect would be considered in $I_{y,servo}$ $I_{y,reg}$ $I_{u,servo}$ $I_{u,reg}$. Integral (19) may be solved by applying the principles of Theorems 1 and 2, and assuming $D_{11}=0$. The results are given by,

$$\lim_{\tau\to\infty} I_{y,pulse} = \mathbf{1}'\mathbb{B}_1'\mathbb{P}_{yp}\mathbb{B}_1\mathbf{1} \tag{30}$$

$$\lim_{\tau\to\infty} I_{u,pulse} = \mathbf{1}'\mathbb{B}_1'\mathbb{P}_{up}\mathbb{B}_1\mathbf{1} \tag{31}$$

where $\mathbb{P}_{yp}$ and $\mathbb{P}_{up}$ are calculated by solving the following Riccati equations,

$$\mathbb{A}'\mathbb{P}_{yp} + \mathbb{P}_{yp}\mathbb{A} = -\mathbb{C}_1'Q\mathbb{C}_1 \tag{32}$$

$$\mathbb{A}'\mathbb{P}_{up}+\mathbb{P}_{up}\mathbb{A}=-\mathbb{C}_2'R\mathbb{C}_2 \tag{33}$$

The quadratic performance indexes deduced in this section complemented the principle of minimum $\phi_{\max}$ (Ruiz-López et al., 2006) as the basis of a design method for optimal and robust MIMO linear controllers. The proposed design method is described in the next section.

3. Controller design method

Summarizing the robustness conditions given by Ruiz-López et al. (2006), and the quadratic performance indexes deduced in Section 2, a linear controller with a suitable robustness and desired performance can be designed from the following optimization problem,

For the system defined with Eqs. (10)–(12) in minimal realization, find elements of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ such that,

$$J(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})=w_1I_{y,servo}+w_2I_{y,reg}+w_3I_{u,servo}+w_4I_{u,reg}+w_5I_{y,pulse}+w_6I_{u,pulse}\rightarrow\min$$

subject to

$$\phi_{\max}=\max\left(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)}\right)<1\ \forall i=1,2,\ldots,(n+k)\quad\text{where } |I_{n+k}\lambda_i-\mathbb{A}|=0$$

$\mathcal{A}$, contains a zero pole

The trade-off between performance and control effort is fitted with weights w_1, w_2, w_3, w_4, w_5 and w_6. This problem can be solved using an adequate search method, like the described in detail by Ruiz-López et al. (2006). Four Riccati equations must be solved in each iteration (two Riccati equations for both step and impulse forcing functions), which is similar to a standard $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control (Doyle et al., 1989). This procedure defines a multiobjective optimal control problem that considers the quadratic performance of error, quadratic performance of control signal, assures the robustness, and does not require to be defined over $\mathcal{H}_2/\mathcal{H}_\infty$ normed spaces, or LMIs. The procedure is illustrated with the design of control systems for two CSTR.

4. Illustrative examples

4.1. A CSTR SISO control with noise

Goncalves et al. (2008) tuned a SISO weighted PID control with noise filter for a CSTR by minimizing a weighted function of $\|\mathbb{T}(s)\|_2$ and $\|\mathbb{T}(s)\|_\infty$. The CSTR and control dynamics were defined with the following transfer functions,

$$c(s)=\frac{500b_0}{s^2+a_1s+a_0}d(s)+\frac{b_0}{s^2+a_1s+a_0}u(s) \tag{34}$$

$$y(s)=c(s)+\eta(s) \tag{35}$$

$$u(s)=k_p\left[\frac{1}{\tau_is}e(s)+e(s)-\frac{\tau_ds}{\rho\tau_ds+1}y(s)\right] \tag{36}$$

where d is a disturbance (exogenous) input, and η is an exogenous disturbance noise. Goncalves et al. (2008) assumed that η is random, uniformly distributed, and $|\eta|\le 0.01$. Supported on the work of Uppal et al. (1974), Goncalves et al. (2008) state three sets of values ($S=\{a_1,a_0,b_0\}$) for the CSTR transfer function, which depend on the CSTR operation point. The three sets, for time unit in seconds are,

$S_1=\{0.01248,5.862,0.03707\}$, $S_2=\{2.674,10.97,0.04107\}$, $S_3=\{9.251,22.19,0.04612\}$

Eqs. (34) and (35) can be represented in state-space form (Eqs. (1) and (2)) with the following matrices,

$$w=\begin{bmatrix}d\\ \eta\end{bmatrix},\quad A=\begin{bmatrix}0&1\\-a_0&-a_1\end{bmatrix},\quad B_1=\begin{bmatrix}0&0\\500b_0&0\end{bmatrix},\quad B_2=\begin{bmatrix}0\\b_0\end{bmatrix},$$
$$C_1=[1\ \ 0],\quad D_{11}=[0\ \ 1],\quad D_{12}=[0], \tag{37}$$

Eq. (36) is the ISA PID configuration, which has weighted PID structure with a noise filter $\rho=1/\mathcal{N}$ (where $\mathcal{N}$ is a noise filtering constant Goncalves et al., 2008), and it can be written in state-space form (Eqs. (3) and (4)) with the following matrices,

$$\mathcal{A}=\begin{bmatrix}0&1\\0&-1/\rho\tau_d\end{bmatrix},\quad \mathcal{B}_1=\begin{bmatrix}\beta_{11}\\ \beta_{21}\end{bmatrix},\quad \mathcal{B}_2=\begin{bmatrix}\beta_{12}\\ \beta_{22}\end{bmatrix},$$
$$\mathcal{C}=[1\ \ 0],\quad \mathcal{D}_1=[\beta_{01}],\quad \mathcal{D}_2=[\beta_{02}] \tag{38}$$

where

$$\beta_{01}=k_p,\quad \beta_{02}=-k_p\left(1+\frac{\tau_d}{\rho\tau_d}\right),\quad \beta_{11}=[-\beta_{01}+k_p(1+\rho\tau_d/\tau_i)]/\rho\tau_d$$

$$\beta_{12}=-[\beta_{02}+k_p(1+\rho\tau_d/\tau_i)]/\rho\tau_d,\quad \beta_{21}=(-\beta_{11}+k_p/\tau_i)/\rho\tau_d,$$
$$\beta_{22}=-(\beta_{12}+k_p/\tau_i)/\rho\tau_d$$

The proposed indexes: Eqs. (21), (23), (24), (25), (30) and (31) with $Q=I=1$, $R=I=1$ (there is only one response and one control signal), $\phi_{\max}$, and $\|\mathbb{T}_{wr,y}(s)\|_\infty$, calculated with the control parameters evaluated by Goncalves et al. (2008) are listed in Table 1.

The proposed method was applied by solving the optimization problem defined in Section 3 for the k_p, τ_i, τ_d, and $\rho=1/\mathcal{N}$ elements of matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ in Eq. (38). The same optimization method described in the first example was used. The model defined by parameters presented in Set 1 (S_1) was selected for the optimization because it has the worst eigenvalues, that is the $\phi_{\max}$ of the open-loop characteristic matrix (A) was the highest among the three sets. The noise filter value ($\mathcal{N}$) was taken between 3 and 10 (Goncalves et al., 2008). The optimization begun from $k_p=12605$, $\tau_i=1.27$, $\tau_d=0.074$, and $\mathcal{N}=9.93$ as initial guess which produces $\phi_{\max}<0.2$. The weights w_i were

Table 1
Quadratic indexes (I) robustness criterion ($\phi_{\max}$) and H_∞-norm obtained for different control parameters of the CSTR.

Method	Control parameters	Indexes, $\phi_{\max}$ and H_∞-norm
Goncalves et al. (2008)	$k_p=7297$, $\tau_i=0.0315$, $\tau_d=0.5772$, $N=10$,	$I_{y,reg}=3.48\times10^{-5}$, $I_{y,pulse}=120$, $I_{u,reg}=2.51\times10^{6}$, $I_{u,pulse}=7.56\times10^{11}$, $I_{y,servo}=0.103$, $I_{u,servo}=3.59\times10^{6}$ $\phi_{\max}=8.24\|\mathbb{T}_{wr,y}(s)\|_\infty=4.47$
Proposed	$k_p=3789$, $\tau_i=0.557$, $\tau_d=0.115$, $N=4.9062$	$I_{y,reg}=5.03\times10^{-3}$, $I_{y,pulse}=16.7$, $I_{u,reg}=2.50\times10^{6}$, $I_{u,pulse}=1.31\times10^{10}$, $I_{y,servo}=0.0814$, $I_{u,servo}=9.21\times10^{5}$ $\phi_{\max}=1\|\mathbb{T}_{wr,y}(s)\|_\infty=1.52$

elected in such way that all the elements of objective function were in the same magnitude order. The final values of these weights were: $w_1 = 5\times10^1$, $w_2 = 1\times10^3$, $w_3 = 2\times10^{-6}$, $w_4 = 2\times10^{-6}$, $w_5 = 5\times10^{-2}$, $w_6 = 3\times10^{-10}$. In Table 1 are listed the optimal results. As in the first illustrative example, a higher value of one performance index, in this case $I_{y,reg}$, was allowed to obtain smaller values of the control signals indexes. In fact, in this example only $I_{y,reg}$ resulted in a greater value than the results of Goncalves et al. (2008). It is important to emphasize that the proposed method control parameters reduced $I_{u,pulse}$ in almost two magnitude orders. Another important result is that the reduction of $\phi_{\max}$ produced a decrease of $\|\mathbb{T}_{wr,y}(s)\|_\infty$. For the performance test, a dynamic simulation of the proposed and Goncalves et al. (2008) controls for the CSTR at the three operation points (S_1, S_2, S_3), was developed applying the random noise $|\eta| \le 0.01$, a unit step in set point, an a unit step disturbance from the fifth second. Simulated output signals are plotted in Fig. 1. As it can be seen in this graph, the proposed method performs with both lower overshot and stabilizing time than the reported by Goncalves et al. (2008) for a set point tracking. The regulatory test, from second 5 onward, show that Goncalves et al. (2008) control performs better than the proposed one as it was expected form the $I_{y,reg}$ results. However, the performance improvement of the proposed method can be better appreciated in Fig. 2, in which the simulated control signals are plotted for the same conditions of Fig. 1, but only the S_1 operation point is shown (the inclusion of the three operations points overload the graph). A drastic reduction of the control signal can be observed with the proposed method, mainly in the reaction to noise. This reduction of control sensitivity to noise was the result of the $I_{u,pulse}$ obtained with the proposed method. These results are very important because in a real operation higher values of control signals could produce saturations. The robustness characteristics of both control design methods can be observed from the fact that the dynamics of the three operations points (S_1, S_2, S_3) are practically even when the differences of some parameters are more than two magnitude orders.

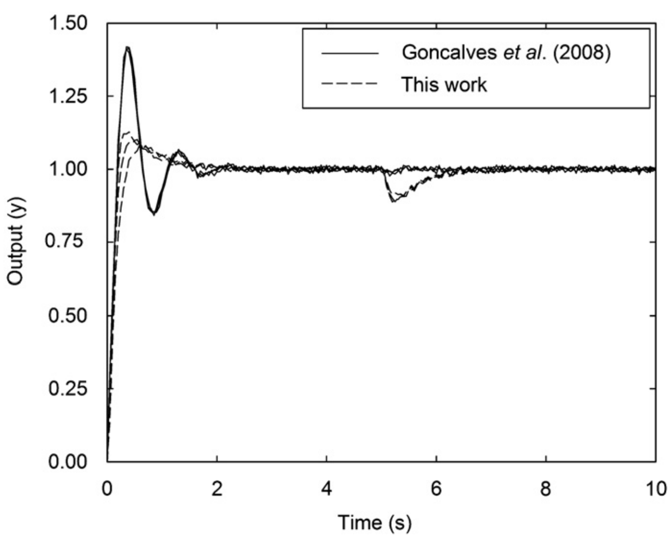

Fig. 1. Dynamic simulation of y for the controlled CSTR as response of unit step in set point r, a unit step disturbance in d applied at 5 s, and random uniformly distributed noise $|\eta| \le 0.01$. Results for the three sets (S_1, S_2, S_3).

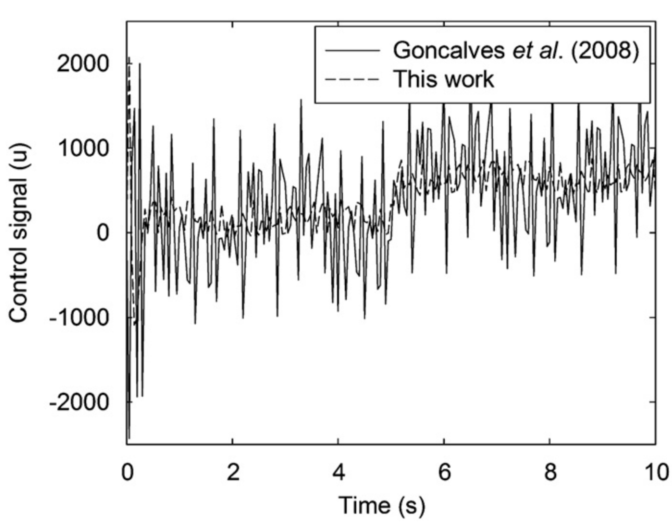

Fig. 2. Dynamic simulation of u for the controlled CSTR as response of unit step in set point r, a unit step disturbance in d applied at 5 s, and random uniformly distributed noise $|\eta| \le 0.01$. Results for set S_1.

4.2. A CSTR MIMO control

Chen et al. (2002) designed via LMIs a MIMO PI control for the CSTR described in the following model,

$$V\frac{dC}{dt} = q(C_f - C) - Vk_0 e^{-E/RT} C \tag{39}$$

$$\rho CpV\frac{dT}{dt} = \rho Cpq(T_f - T) + (-\Delta H)Vk_0 e^{-E/RT} C + \rho_c Cp_c q_c (1 - e^{-h/\rho_c Cp_c q_c})(T_{cf} - T) \tag{40}$$

Eqs. (39) and (40) represent a non-linear dynamic system that may have multiple steady states, and in which the inputs variables may be bounded. In order to apply LMIs, Chen et al. (2002) linearized Eqs. (39) and (40) by applying Taylor series expansion around the neighborhoods of a given steady state. The obtained model may be represented in linear space state form (Eqs. (1) and (2)) with the following matrices,

$$x = \begin{bmatrix} C - C_s \\ T - T_s \end{bmatrix}, \quad w = \begin{bmatrix} C_f - C_{fs} \\ T_f - T_{fs} \end{bmatrix}, \quad u = \begin{bmatrix} q - q_s \\ q_c - q_{cs} \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad y = \begin{bmatrix} C - C_s \\ T - T_s \end{bmatrix}, \quad r = \begin{bmatrix} C_d - C_s \\ T_d - T_s \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The nominal steady state was reported as $C_s = 0.1\,\text{mol}\,\text{L}^{-1}$, $T_s = 438.54\,\text{K}$, $q_s = 100\,\text{L}\,\text{min}^{-1}$, $q_{cs} = 103.41\,\text{L}\,\text{min}^{-1}$, $C_{fs} = 1\,\text{mol}\,\text{L}^{-1}$, $T_{fs} = 350\,\text{K}$, $T_{cf} = 350\,\text{K}$, $V = 100\,\text{L}$, $h = 7\times10^5\,\text{cal}\,\text{min}^{-1}\,\text{K}^{-1}$, $k_0 = 7.2\times10^{10}\,\text{L}\,\text{min}^{-1}$, $E/R = 10^4\,\text{K}$, $\Delta H = -2\times10^5\,\text{cal}\,\text{mol}^{-1}$, $\rho = \rho_c = 10^3\,\text{g}\,\text{L}^{-1}$, $Cp = Cp_c = 1\,\text{cal}\,\text{g}^{-1}\,\text{K}^{-1}$. At this steady state, the set of parameters, $S = \{a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{12}, b_{22}\}$ is given by

$$S_0 = \{-9.999, -0.0468, 1799.8, 7.328, 0.009, 0, -0.885, -0.878\}$$

However, Chen et al. (2002) used a set of parameters that represent the average process behavior between the limits of operations ranges. This set of parameters is

$$S_1 = \{-14.677, -0.0453, 2735.3, 6.978, 0.00858, 0, -0.885, -0.867\}$$

Table 2
Quadratic indexes (I) robustness criterion ($\phi_{\max}$) and H_∞-norm obtained for different control parameters of the MIMO CSTR.

Method	Control parameters	Indexes, $\phi_{\max}$ and H_∞-norm
Chen et al. (2002)	$k_{p11}=5783.1$, $k_{p12}=27.816$ $k_{p21}=-8724$, $k_{p22}=-137.69$ $k_{i11}=63\,687$, $k_{i12}=252.68$ $k_{i21}=-30\,645$, $k_{i22}=-1129.1$	$I_{y,reg}=3.16\times10^{-8}$, $I_{y,pulse}=1.89\times10^{-5}$, $I_{u,reg}=4.29\times10^{1}$, $I_{u,pulse}=3.36\times10^{3}$, $I_{y,servo}=4.22\times10^{-5}$, $I_{u,servo}=9.04\times10^{3}$ $\phi_{\max}=0\Vert\mathbb{T}_{wr,y}(s)\Vert_\infty=25.9$
Ruiz-López et al. (2006)	$k_{p11}=7732.5$, $k_{p12}=0$ $k_{p21}=-13\,771$, $k_{p22}=-673.64$ $k_{i11}=93\,938.6$, $k_{i12}=142.6$ $k_{i21}=-97\,507$, $k_{i22}=-6634.2$	$I_{y,reg}=1.91\times10^{-8}$, $I_{y,pulse}=1.55\times10^{-5}$, $I_{u,reg}=4.28\times10^{1}$, $I_{u,pulse}=4.95\times10^{3}$, $I_{y,servo}=2.07\times10^{-5}$, $I_{u,servo}=1.06\times10^{4}$ $\phi_{\max}=0\Vert\mathbb{T}_{wr,y}(s)\Vert_\infty=8.23$
Proposed	$k_{p11}=7733.6$, $k_{p12}=1.2$ $k_{p21}=-13\,772$, $k_{p22}=-674.5$ $k_{i11}=93\,948$, $k_{i12}=141.7$ $k_{i21}=-97\,506$, $k_{i22}=-6633.1$ $k_{d11}=126.1$, $k_{d12}=14.9$ $k_{d21}=-801.2$, $k_{d22}=-9002.1$ $\tau_{d1}=49.8$, $\tau_{d2}=48.5$	$I_{y,reg}=1.85\times10^{-8}$, $I_{y,pulse}=1.58\times10^{-5}$, $I_{u,reg}=3.87\times10^{1}$, $I_{u,pulse}=4.66\times10^{3}$, $I_{y,servo}=1.80\times10^{-5}$, $I_{u,servo}=6.66\times10^{3}$ $\phi_{\max}=0\Vert\mathbb{T}_{wr,y}(s)\Vert_\infty=6.57$

The MIMO PI algorithm can be written in space state form (Eqs. (3) and (4)) with the following matrices,

$$\mathcal{A}=\begin{bmatrix}0&0\\0&0\end{bmatrix},\quad \mathcal{B}_1=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \mathcal{B}_2=\begin{bmatrix}-1&0\\0&-1\end{bmatrix},$$
$$\mathcal{C}=\begin{bmatrix}k_{i11}&k_{i12}\\k_{i21}&k_{i22}\end{bmatrix},\quad \mathcal{D}_1=\begin{bmatrix}k_{p11}&k_{p12}\\k_{p21}&k_{p22}\end{bmatrix},\quad \mathcal{D}_2=\begin{bmatrix}-k_{p11}&-k_{p12}\\-k_{p21}&-k_{p22}\end{bmatrix}$$

Chen et al. (2002) obtained the set of control parameters listed in Table 2 by applying LMIs and a space state representation with uncertain limits. The proposed indexes: Eqs. (21), (23), (24), (25), (30) and (31), $\phi_{\max}$, and $\Vert\mathbb{T}_{wr,y}(s)\Vert_\infty$ calculated with parameter set S_1 are also listed in Table 2. Reported values were evaluated with the following weight and input matrices,

$$Q=\begin{bmatrix}1&0\\0&1\times10^{-4}\end{bmatrix},\quad R=I=1,\quad w(s)=\begin{bmatrix}0.05\\1\end{bmatrix}s^{-1},\quad r(s)=\begin{bmatrix}0.05\\1\end{bmatrix}s^{-1} \tag{41}$$

The election of matrices Q and R was done by considering that output concentration (C) varies in the magnitude order of 10^{-1} and temperature (T) changes occur in the magnitude order of 10^{1}; while variations in both control variables (q and q_c) have the same magnitude order.

For the same system, Ruiz-López et al. (2006) calculated another sets of parameters, by applying a minimization of both $\phi_{\max}$ and the quadratic performance index of Eq. (6) with $R=0$, $r=0$, $w=0$ and $x(0)=x_0\neq0$. The control parameters with their corresponding indexes (calculated with set S_1, and the same weight and input matrices of Eq. (41)) are listed in Table 2. As it can be observed, the control parameters reported by Ruiz-López et al. (2006) produce a reduction in all indexes with the exception of $I_{u,pulse}$ and $I_{u,servo}$. The performance of both control systems under a simultaneous change of set point in concentration and temperature to $C_d=0.09\,\mathrm{mol\,L^{-1}}$ and $T_d=433.54\,\mathrm{K}$, followed by a simultaneous step change in exogenous inputs to $C_f=0.90\,\mathrm{mol\,L^{-1}}$ and $T_f=345\,\mathrm{K}$ at minute 2 are plotted in Figs. 3–6. It is important to emphasize that control performance was simulated directly from the non-linear description of reactor (Eqs. (39) and (40)) with linear control equations (Eqs. (3) and (4)), and the control variables, that is the input and cooling flows (q and q_c), were bounded between $\pm40\,\mathrm{L\,min^{-1}}$ of their nominal values. The plotted behavior shows the effect of reducing $I_{y,reg}$ which is manifested through a lower overshoot in output

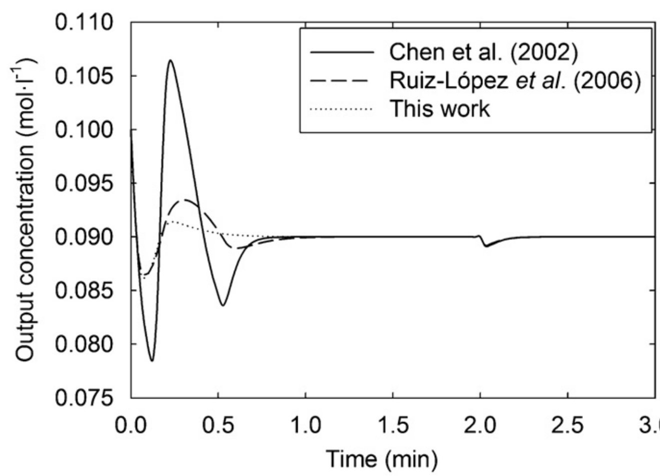

Fig. 3. Dynamic simulation of output concentration for the controlled CSTR as response of simultaneous step in set points concentration and temperature, followed by a simultaneous step in input concentration and temperature.

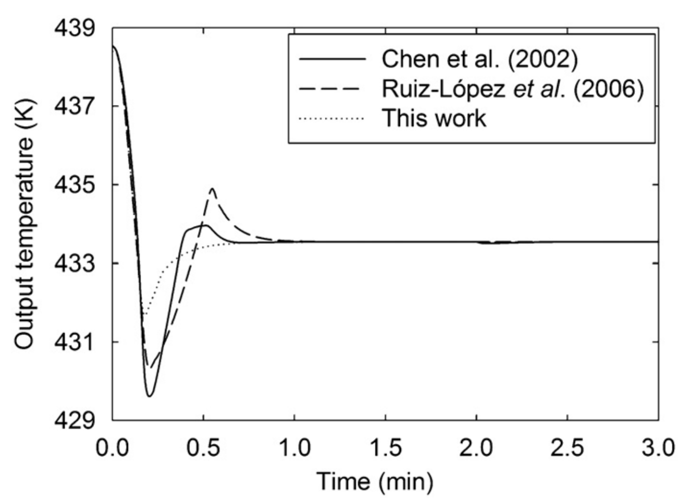

Fig. 4. Dynamic simulation of output temperature for the controlled CSTR as response of simultaneous step in set points concentration and temperature, followed by a simultaneous step in input concentration and temperature.

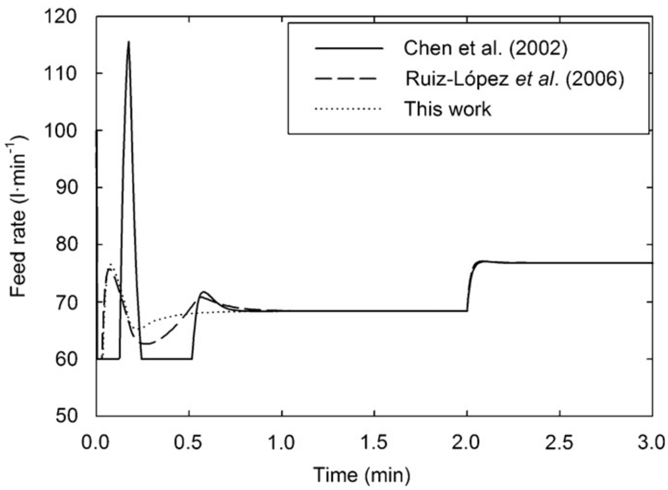

Fig. 5. Dynamic simulation of feed flow for the controlled CSTR as response of simultaneous step in set points concentration and temperature, followed by a simultaneous step in input concentration and temperature.

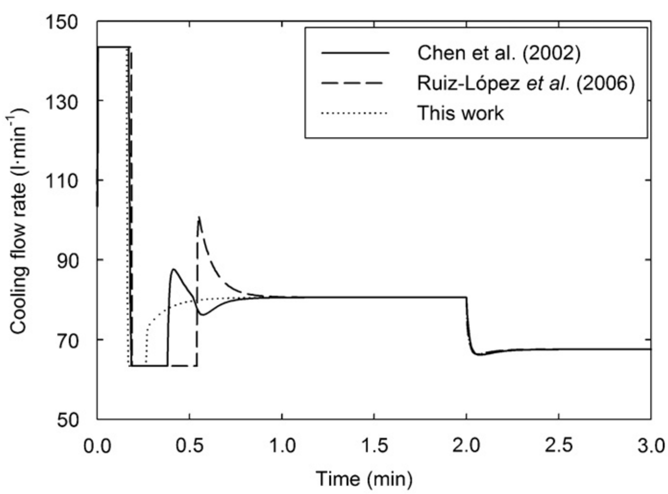

Fig. 6. Dynamic simulation of cooling flow for the controlled CSTR as response of simultaneous step in set points concentration and temperature, followed by a simultaneous step in input concentration and temperature.

concentration (Fig. 3), while the reduction of $I_{u,reg}$, is reflected in a smaller change of the feed flow (Fig. 5). This last effect produces a lower decrease of output temperature (Fig. 4).

An optimization search showed that there is not a better set of control parameters simultaneously minimizing all the indexes. That is, in this particular case the control parameters reported by Ruiz-López et al. (2006) are optimal for a MIMO PI algorithm. However, one of the main advantages of the proposed method is that performance indexes can be easily calculated for any linear control algorithm. Therefore, the ISA PID algorithm (Eq. (36)) was implemented for this MIMO system in order to improve the performance of the controller reported by Ruiz-López et al. (2006). A MIMO ISA PID algorithm can be represented by Eqs. (3) and (4) with the following matrices,

$$\mathcal{A}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -1/\tau_1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -1/\tau_2\end{bmatrix},\quad \mathcal{B}_1=\begin{bmatrix}\beta_{11}^{11} & \beta_{12}^{11}\\ \beta_{11}^{21} & \beta_{12}^{21}\\ \beta_{21}^{11} & \beta_{22}^{11}\\ \beta_{21}^{21} & \beta_{22}^{21}\end{bmatrix},$$

$$\mathcal{B}_2=\begin{bmatrix}\beta_{11}^{12} & \beta_{12}^{12}\\ \beta_{11}^{22} & \beta_{12}^{22}\\ \beta_{21}^{12} & \beta_{22}^{12}\\ \beta_{21}^{22} & \beta_{22}^{22}\end{bmatrix},\quad \mathcal{C}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix},$$

$$\mathcal{D}_1=\begin{bmatrix}\beta_{11}^{01} & \beta_{12}^{01}\\ \beta_{21}^{01} & \beta_{22}^{01}\end{bmatrix},\quad \mathcal{D}_2=\begin{bmatrix}\beta_{11}^{02} & \beta_{12}^{02}\\ \beta_{21}^{02} & \beta_{22}^{02}\end{bmatrix},$$

$$\beta_{ij}^{01}=k_{pij},\quad \beta_{ij}^{02}=-\left(k_{pij}+\frac{k_{dij}}{\tau_i}\right),\quad \beta_{ij}^{11}=[-\beta_{ij}^{01}+k_{pij}+k_{iij}\tau_i]/\tau_i$$

$$\beta_{ij}^{12}=-[\beta_{ij}^{02}+k_{pij}+k_{iij}\tau_i]/\tau_i,\quad \beta_{ij}^{21}=(-\beta_{ij}^{11}+k_{iij})/\tau_i,$$
$$\beta_{ij}^{22}=-(\beta_{ij}^{12}+k_{iij})/\tau_i$$

An optimization search with weight factors $w_1=0.5\times10^8$, $w_2=1\times10^5$, $w_3=0.25\times10^{-1}$, $w_4=0.2\times10^{-3}$, $w_5=0.5\times10^5$, $w_6=1\times10^{-4}$ produced the parameter set and performance indexes listed in Table 2. A general reduction in all indexes was obtained in comparison with those calculated for the control parameters reported by Ruiz-López et al. (2006) and Chen et al. (2002). The control performance (direct simulation of Eqs. (39) and (40)) plotted in Figs. 3–6 shows an improvement for both output dynamics (Figs. 3 and 4) and control signals (Figs. 5 and 6). In general, the dynamic behavior of the MIMO ISA PID controller for the proposed model performs with lower overshoots in outputs and smaller changes in control signals. It is important to note that because simulations were performed with the original nonlinear model and with the control signal bounded, the improvement in both tracking performance and disturbance rejection demonstrates the robustness characteristics of the proposed controller.

5. Conclusion

The integrals of squared error and squared control signal deduced in this paper complete the robustness concept proposed by Ruiz-López et al. (2006) as theoretical basis for a design method of quadratic optimal and robust MIMO linear controllers. The design method was stated as a minimization of quadratic performance indexes subjected to a constraint in the $\phi_{\max}$ ratio. The presented method, applied in the design of both SISO and MIMO controllers for two CSTR, showed an improvement with respect to the previous LMI-based controllers defined in $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$ normed spaces. Therefore, the presented results demonstrate the advantages of the proposed method to develop robust controllers with good performance characteristics for both the servomechanism and regulator problems.

Acknowledgments

The authors express their acknowledges with Mexican DGEST by the financial support through the project entitle "Control de sistemas no-lineales utilizando algoritmos de control lineales".

Appendix A. Proof of Theorem 1

If the system (10)–(12) is in minimal realization $\mathbb{A}$ is stable, and the proposed controller contains a zero pole (integral action), under a step disturbance in $r(t)$, there is not off set ($e(t)\in\mathcal{L}_{2+}$) and therefore,

$$\lim_{t\to\infty} y(t)=r(t)=K_2 \tag{A.1}$$

Eq. (16) for the servomechanism problem ($K_1=0$) is,

$$y=\mathbb{C}_1\mathbb{A}^{-1}e^{\mathbb{A}t}\mathbb{B}_2K_2-\mathbb{C}_1\mathbb{A}^{-1}\mathbb{B}_2K_2+\mathbb{D}_{12}K_2 \tag{A.2}$$

Due the system is in minimal realization and $\mathbb{A}$ is stable,

$$\lim_{t\to\infty}\mathbb{C}_1\mathbb{A}^{-1}e^{\mathbb{A}t}\mathbb{B}_2K_2=0 \tag{A.3}$$

(A.1)–(A.3) implies,

$$-\mathbb{C}_1\mathbb{A}^{-1}\mathbb{B}_2+\mathbb{D}_{12}=I_r \tag{A.4}$$

Therefore applying (A.2) and (A.4),

$$\begin{aligned} I_{y,servo} &= \int_0^\tau (K_2-y)'Q(K_2-y)\,dt \\ &= \int_0^\tau (\mathbb{C}_1\mathbb{A}^{-1}e^{\mathbb{A}t}\mathbb{B}_2K_2)'Q\mathbb{C}_1\mathbb{A}^{-1}e^{\mathbb{A}t}\mathbb{B}_2K_2\,dt \end{aligned} \tag{A.5}$$

or

$$I_{y,servo}=K_2'\mathbb{B}_2'\int_0^\tau (e^{\mathbb{A}t})'(\mathbb{A}^{-1})'\mathbb{C}_1'Q\mathbb{C}_1\mathbb{A}^{-1}e^{\mathbb{A}t}dt\mathbb{B}_2K_2 \tag{A.6}$$

defining

$$\Theta=e^{\mathbb{A}t} \tag{A.7}$$

(A.7) is the solution of the following differential equation and initial condition,

$$\dot{\Theta}=\mathbb{A}\Theta, \quad \Theta(0)=\mathbf{1} \tag{A.8}$$

And, like $\mathbb{A}$ is stable, exist a Lyapunov function such that,

$$V(t)=\Theta^T\mathbb{P}_y\Theta \tag{A.9}$$

where $\mathbb{P}_y$ can be calculating from the Riccati equation,

$$\mathbb{A}'\mathbb{P}_y+\mathbb{P}_y\mathbb{A}=-(\mathbb{A}^{-1})'\mathbb{C}_1'Q\mathbb{C}_1\mathbb{A}^{-1} \tag{A.10}$$

Then, by the properties of a Lyapunov function in Eq. (A.6),

$$\int_0^\tau \Theta'(\mathbb{A}^{-1})'\mathbb{C}_1'Q\mathbb{C}_1\mathbb{A}^{-1}\Theta\,dt=-V(t)|_0^\tau=-\Theta'(\tau)\mathbb{P}_y\Theta(\tau)+\Theta'(0)\mathbb{P}_y\Theta(0) \tag{A.11}$$

Finally, due the system is in minimal realization and $\mathbb{A}$ is stable,

$$\lim_{\tau\to\infty}\Theta'(\tau)\mathbb{P}_y\Theta(\tau)=0$$

and therefore from Eqs. (A.6), (A.8) and (A11),

$$\lim_{\tau\to\infty}I_{y,servo}=K_2'\mathbb{B}_2'\mathbb{P}_y\mathbb{B}_2K_2$$

which proof the Theorem 1.□

Appendix B. Proof of Theorem 2

If the system (10)–(12) is in minimal realization $\mathbb{A}$ is stable, and the proposed controller contains a zero pole (integral action), under a step disturbance in $w(t)$, $e(t)\in\mathcal{L}_{2+}$ and therefore,

$$\lim_{t\to\infty}y(t)=0, \quad (\text{for } K_2=0) \tag{B.1}$$

then

$$y=\mathbb{C}_1\mathbb{A}^{-1}e^{\mathbb{A}t}\mathbb{B}_1K_1-\mathbb{C}_1\mathbb{A}^{-1}\mathbb{B}_1K_1+\mathbb{D}_{11}K_1 \tag{B.2}$$

(B.1) implies,

$$-\mathbb{C}_1\mathbb{A}^{-1}\mathbb{B}_1+\mathbb{D}_{11}=0 \tag{B.3}$$

under this considerations,

$$I_{y,reg}=K_1'\mathbb{B}_1'\int_0^\tau (e^{\mathbb{A}t})'(\mathbb{A}^{-1})'\mathbb{C}_1'Q\mathbb{C}_1\mathbb{A}^{-1}e^{\mathbb{A}t}\,dt\mathbb{B}_1K_1 \tag{B.4}$$

And therefore by the Theorem 1 the Theorem 2 is proofed.□

References

Alvarez, J., Alvarez, J., González, E., 1989. Global nonlinear control of a continuous stirred tank reactor. Chemical Engineering Science 44, 1147–1160.
Alvarez, J., Morales, A., Cervantes, I., 1998. Robust proportional-integral control. Industrial & Engineering Chemistry Research 37, 4740–4747.
Bao, J., Forbes, J.F., McLellan, P.J., 1999. Robust multiloop PID controller design: a successive semidefinite programming approach. Industrial & Engineering Chemistry Research 38, 3407–3419.
Chen, C.L., Wang, T.C., Hsu, S.H., 2002. An LMI approach to H PI controller design. Journal of Chemical Engineering of Japan 35, 83–93.
Cheng, C., Chiu, M.S., 2008. Robust controller design for nonlinear processes using JITL technique. Chemical Engineering Science 63, 5141–5148.
Chilali, M., Gahinet, P., Apkarian, P., 1999. Robust pole placement in LMI regions. IEEE Transactions on Automatic Control 44, 2257–2270.
Doyle, J.C., Glover, K., Khargonekar, P.P., Francis, B.A., 1989. State-space solutions to standard H_2 and H_∞ control problems. IEEE Transactions on Automatic Control 34, 831–845.
El-Kady, M.M., Salim, M.S., El-Sagheer, A.M., 2003. Numerical treatment of multiobjective optimal control problems. Automatica 39, 47–55.
Engwerda, J., Weeren, A., 2008. A result on output feedback linear quadratic control. Automatica 44, 265–271.
García-Alvarado, M.A., Ruiz-López, I.I., Torres-Ramos, T., 2005. Tuning of multivariate PID controllers based on characteristic matrix eigenvalues, Lyapunov functions and robustness criteria. Chemical Engineering Science 60, 897–905.
Ge, M., Chiu, M.S., Wang, Q.G., 2002. Robust PID controller design via LMI approach. Journal of Process Control 12, 3–13.
Goncalves, E.N., Palhares, R.M., Takahashi, R.H.C., 2008. A novel approach for H_2/H_∞ robust PID synthesis for uncertain systems. Journal of Process Control 18, 19–26.
Ruiz-López, I.I., Rodríguez-Jimenes, G.C., García-Alvarado, M.A., 2006. Robust MIMO PID controllers tuning based on complex/real ratio of the characteristic matrix eigenvalues. Chemical Engineering Science 61, 4332–4340.
Saeki, M., 2006. Fixed structure PID controller design for standard H_∞ control problem. Automatica 42, 93–100.
Tan, K.K., Ferdous, R., Huang, S., 2002a. Closed-loop automatic tuning of PID controller for nonlinear systems. Chemical Engineering Science 57, 3005–3011.
Tan, W., Marquez, H.J., Chen, T., 2002b. Multivariable robust controller design for a boiler system. IEEE Transactions on Control Systems Technology 10, 735–742.
Toscano, R., 2005. A simple robust PI/PID controller design via numerical optimization approach. Journal of Process Control 15, 81–88.
Uppal, U., Ray, W.H., Poore, A.B., 1974. On the dynamic behavior of continuous stirred tank reactors. Chemical Engineering Science 29, 967–985.
Xiong, Q., Cai, W.J., He, M.J., 2007. Equivalent transfer function method for PI/PID controller design of MIMO processes. Journal of Process Control 17, 665–673.