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## Applied Mathematical Modelling

journal homepage: [www.elsevier.com/locate/apm](http://www.elsevier.com/locate/apm)

## Short communication

Combined state and least squares parameter estimation algorithms for dynamic systems<sup>☆</sup>

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## ARTICLE INFO

## Article history:

Received 15 January 2013

Accepted 1 June 2013

Available online xxxx

## Keywords:

Dynamic system  
 Numerical algorithm  
 Least squares  
 Parameter estimation  
 Recursive identification  
 State space model

## ABSTRACT

The control theory and automation technology cast the glory of our era. Highly integrated computer chip and automation products are changing our lives. Mathematical models and parameter estimation are basic for automatic control. This paper discusses the parameter estimation algorithm of establishing the mathematical models for dynamic systems and presents an estimated states based recursive least squares algorithm, and the states of the system are computed through the Kalman filter using the estimated parameters. A numerical example is provided to confirm the effectiveness of the proposed algorithm.

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## 1. Introduction

Numerical methods have wide applications for solving matrix equations or compute the model parameters of dynamic systems [1–3]. Typical numerical identification methods include the gradient search, the least squares and the Newton methods [4–6]. Parameter estimation is basic for controller design [7–9], filtering and state estimation [10,11] and system identification [12–14]. Recently, a gradient based iterative method and a least squares based iterative method were presented for identifying multiple-input multiple-output systems [15] and for identifying Wiener nonlinear systems [16]; and a Newton recursive and a Newton iterative algorithms were developed for identifying Hammerstein nonlinear systems [17]; a least squares based recursive estimation algorithm and a least squares based iterative algorithm were proposed for output error moving average systems using data filtering [18]; several maximum likelihood based recursive least squares algorithms were discussed for systems with colored noises [19–21].

In the area of parameter estimation [22–25], Zhang et al. proposed a bias compensation based recursive least squares method for stochastic systems with colored noises [26] and for a class of multiple-input single-output systems [27]; Liu et al. discussed multi-innovation stochastic gradient approach for multiple-input single-output systems using the multi-innovation identification theory and the auxiliary model identification idea [28] and analyzed the convergence of the stochastic gradient algorithm for multivariable ARX-like systems [29]. Ding et al. presented an auxiliary model based multi-innovation stochastic gradient algorithm for systems with scarce measurements [30] and an auxiliary model based recursive least squares algorithm for missing-data systems [31]. Xiao et al. presented a residual based interactive least squares

\* This work was supported by the National Natural Science Foundation of China (No. 61273194), the Natural Science Foundation of Jiangsu Province (China, BK2012549), the 111 Project (B12018) and the PAPD of Jiangsu Higher Education Institutions.

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algorithm for controlled autoregressive moving average systems [32]; Ding and Duan proposed a two-stage parameter estimation algorithms for Box-Jenkins systems [33].

In the field of state space system identification, Ding et al. presented a hierarchical identification method for the lifted state space model of general dual-rate systems [34] and for non-uniformly sampled-data systems [35]; Gu et al. discussed a least squares numerical parameter estimation algorithm for a state space model with multi-state delays, assuming the states of the system are available [36], and studied parameter and state estimation for a state space model with a one-unit state delay [37] and for a multivariable state space system with d-step state-delay [38]. This paper studied the identification method of canonical state space systems, assuming the states of the system are unavailable.

This paper is organized as follows. Section 2 derives the identification model for state space systems. Section 3 gives the parameter and state estimation algorithm. Section 4 provides an example to verify the effectiveness of the proposed algorithm. Finally, concluding remarks are given in Section 5.

## 2. The identification model for the state space systems

Let us define some notations. “ $A := X$ ” or “ $X := A$ ” stands for “ $A$  is defined as  $X$ ”. Let  $z$  denote a unit forward shift operator with  $z\mathbf{x}(t) = \mathbf{x}(t+1)$  and  $z^{-1}\mathbf{x}(t) = \mathbf{x}(t-1)$ .

Consider the following observer canonical state space system,

$$\mathbf{x}(t+1) = \mathbf{Ax}(t) + \mathbf{bu}(t), \quad (1)$$

$$y(t) = \mathbf{cx}(t) + v(t), \quad (2)$$

where  $\mathbf{x}(t) := [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}$  is the system input,  $y(t) \in \mathbb{R}$  is the system output,  $v(t) \in \mathbb{R}$  is random noise with zero mean,  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{b} \in \mathbb{R}^n$  and  $\mathbf{c}$  are the system parameter matrix and vectors:

$$\mathbf{A} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ -a_{n-1} & 0 & \cdots & 0 & 1 \\ -a_n & 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad \mathbf{b} := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} \in \mathbb{R}^n,$$

$$\mathbf{c} := [1, 0, 0, \dots, 0] \in \mathbb{R}^{1 \times n}.$$

The parameters  $a_i \in \mathbb{R}$  and  $b_i \in \mathbb{R}$  are to be identified from observation data  $\{u(t), y(t) : t = 1, 2, 3, \dots\}$ .

From (1), we have

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_{n-1}(t+1) \\ x_n(t+1) \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ -a_{n-1} & 0 & \cdots & 0 & 1 \\ -a_n & 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u(t), \quad (3)$$

$$y(t) = [1, 0, 0, \dots, 0]\mathbf{x}(t) + v(t), \quad (4)$$

which can be written as

$$x_i(t+1) = -a_i x_1(t) + x_{i+1}(t) + b_i u(t), \quad i = 1, 2, \dots, (n-1), \quad (5)$$

$$x_n(t+1) = -a_n x_1(t) + b_n u(t), \quad (6)$$

$$y(t) = x_1(t) + v(t). \quad (7)$$

Multiplying (5) by  $z^{-i}$  gives

$$x_i(t-i+1) = -a_i x_1(t-i) + x_{i+1}(t-i) + b_i u(t-i), \quad i = 1, 2, \dots, (n-1).$$

Summing for  $i$  from  $i = 1$  to  $i = (n-1)$  gives

$$\sum_{i=1}^{n-1} x_i(t-i+1) = -\sum_{i=1}^{n-1} a_i x_1(t-i) + \sum_{i=1}^{n-1} x_{i+1}(t-i) + \sum_{i=1}^{n-1} b_i u(t-i),$$

or

$$x_1(t) = -\sum_{i=1}^{n-1} a_i x_1(t-i) + x_n(t-n+1) + \sum_{i=1}^{n-1} b_i u(t-i). \quad (8)$$

Multiplying (6) by  $z^{-n}$  gives

$$x_n(t-n+1) = -a_n x_1(t-n) + b_n u(t-n). \quad (9)$$

Substituting (9) into (8) gives

$$x_1(t) = -\sum_{i=1}^{n-1} a_i x_1(t-i) - a_n x_1(t-n) + b_n u(t-n) + \sum_{i=1}^{n-1} b_i u(t-i) = -\sum_{i=1}^n a_i x_1(t-i) + \sum_{i=1}^n b_i u(t-n). \quad (10)$$

Define the parameter vector  $\theta$  and the information vector  $\phi(t)$  as

$$\theta := \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \in \mathbb{R}^{2n}, \quad \phi(t) := \begin{bmatrix} \phi(t) \\ \psi(t) \end{bmatrix} \in \mathbb{R}^{2n},$$

$$\mathbf{a} := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \quad \mathbf{b} := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n, \quad \phi(t) := \begin{bmatrix} -x_1(t-1) \\ -x_1(t-2) \\ \vdots \\ -x_1(t-n) \end{bmatrix} \in \mathbb{R}^n, \quad \psi(t) := \begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(t-n) \end{bmatrix} \in \mathbb{R}^n.$$

Using (10) and from (7), we obtain the identification model of the state space system in (1) and (2):

$$y(t) = x_1(t) + v(t) = \phi^\top(t)\mathbf{a} + \psi^\top(t)\mathbf{b} + v(t) = [\phi^\top(t), \psi^\top(t)] \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} + v(t)\phi^\top(t)\theta + v(t). \quad (11)$$

The information vector  $\phi(t)$  consists of the state  $x_1(t-i)$  and the input  $u(t-i)$ , and the parameter vector  $\theta$  consists of all the parameters  $a_i$  and  $b_i$  of the state space system in (1).

### 3. The parameter and state estimation algorithm

#### 3.1. The state estimation algorithm

If the parameter matrix/vector  $\mathbf{A}$  and  $\mathbf{b}$  are known, then we can apply the following Kalman filter to generate the estimate  $\hat{x}(t)$  of the state vector  $x(t)$ :

$$\hat{x}(t+1) = \mathbf{A}\hat{x}(t) + \mathbf{b}u(t) + \mathbf{L}_1(t)[y(t) - \mathbf{c}\hat{x}(t)], \quad \hat{x}(1) = \mathbf{1}_n/p_0, \quad (12)$$

$$\mathbf{L}_1(t) = \mathbf{A}\mathbf{P}_1(t)\mathbf{c}^\top[1 + \mathbf{c}\mathbf{P}_1(t)\mathbf{c}^\top]^{-1}, \quad (13)$$

$$\mathbf{P}_1(t+1) = \mathbf{A}\mathbf{P}_1(t)\mathbf{A}^\top - \mathbf{L}_1(t)\mathbf{c}\mathbf{P}_1(t)\mathbf{c}^\top, \quad \mathbf{P}_1(1) = \mathbf{I}_n. \quad (14)$$

When the parameter matrix/vector  $\mathbf{A}$  and  $\mathbf{b}$  are unknown, then we use the estimated parameter vector

$$\hat{\theta}(t) = [\hat{a}_1(t), \hat{a}_2(t), \dots, \hat{a}_n(t), \hat{b}_1(t), \hat{b}_2(t), \dots, \hat{b}_n(t)]^\top,$$

to construct the estimates  $\hat{\mathbf{A}}(t)$  and  $\hat{\mathbf{b}}(t)$  of  $\mathbf{A}$  and  $\mathbf{b}$  and use the estimated parameter matrix  $\hat{\mathbf{A}}(t)$  and the parameter vector  $\hat{\mathbf{b}}(t)$  to compute the estimate  $\hat{x}(t)$  of the state vector  $x(t)$  [37,38]:

$$\hat{x}(t+1) = \hat{\mathbf{A}}(t)\hat{x}(t) + \hat{\mathbf{b}}(t)u(t) + \mathbf{L}_2(t)[y(t) - \mathbf{c}\hat{x}(t)], \quad \hat{x}(1) = \mathbf{1}_n/p_0, \quad (15)$$

$$\mathbf{L}_2(t) = \hat{\mathbf{A}}(t)\mathbf{P}_2(t)\mathbf{c}^\top[1 + \mathbf{c}\mathbf{P}_2(t)\mathbf{c}^\top]^{-1}, \quad (16)$$

$$\mathbf{P}_2(t+1) = \hat{\mathbf{A}}(t)\mathbf{P}_2(t)\hat{\mathbf{A}}^\top(t) - \mathbf{L}_2(t)\mathbf{c}\mathbf{P}_2(t)\hat{\mathbf{A}}^\top(t), \quad \mathbf{P}_2(1) = \mathbf{I}_n, \quad (17)$$

$$\hat{\mathbf{A}}(t) = \begin{bmatrix} -\hat{a}_1(t) & 1 & 0 & \cdots & 0 \\ -\hat{a}_2(t) & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ -\hat{a}_{n-1}(t) & 0 & \cdots & 0 & 1 \\ -\hat{a}_n(t) & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad \hat{\mathbf{b}}(t) = \begin{bmatrix} \hat{b}_1(t) \\ \hat{b}_2(t) \\ \vdots \\ \hat{b}_{n-1}(t) \\ \hat{b}_n(t) \end{bmatrix}. \quad (18)$$

#### 3.2. The parameter estimation algorithm

Let  $\hat{\theta}(t)$  represent the estimate of  $\theta$  at time  $t$ . According to the least squares principle, defining and minimizing the quadratic criterion function

$$J(\theta) := \sum_{j=1}^t [y(j) - \phi^T(j)\theta]^2,$$

we can obtain the following recursive algorithm:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + P(t)\phi(t)[y(t) - \phi^T(t)\hat{\theta}(t-1)], \quad \hat{\theta}(0) = \mathbf{1}_{2n}/p_0, \quad (19)$$

$$P^{-1}(t) = P^{-1}(t-1) + \phi(t)\phi^T(t), \quad P(0) = p_0 I_{2n}, \quad (20)$$

where  $\mathbf{1}_{2n}$  denotes a  $2n$ -dimensional column vector whose elements are all unity,  $p_0$  is generally taken to be a large positive number, e.g.,  $p_0 = 10^6$ .

Because the information vector  $\phi(t)$  contains the unmeasurable state variable  $x_1(t-i)$  in  $\phi(t)$ , the algorithm in (19) and (20) is impossible to implement. The scheme here is to replace  $x_1(t-i)$  in  $\phi(t)$  with its estimated state  $\hat{x}_1(t-i)$  and to define

$$\hat{\phi}(t) := \begin{bmatrix} \hat{\phi}(t) \\ \psi(t) \end{bmatrix} \in \mathbb{R}^{2n}, \quad \hat{\phi}(t) := \begin{bmatrix} -\hat{x}_1(t-1) \\ -\hat{x}_1(t-2) \\ \vdots \\ -\hat{x}_1(t-n) \end{bmatrix} \in \mathbb{R}^n. \quad (21)$$

Replacing  $\phi(t)$  in (19) and (20) with its estimate  $\hat{\phi}(t)$  yields

$$\hat{\theta}(t) = \hat{\theta}(t-1) + P(t)\hat{\phi}(t)[y(t) - \hat{\phi}^T(t)\hat{\theta}(t-1)], \quad \hat{\theta}(0) = \mathbf{1}_{2n}/p_0, \quad (22)$$

$$P^{-1}(t) = P^{-1}(t-1) + \hat{\phi}(t)\hat{\phi}^T(t), \quad P(0) = p_0 I_{2n}. \quad (23)$$

Applying the matrix inversion lemma [1,36]

$$(A + BC)^{-1} = A^{-1} - A^{-1}B(I + CA^{-1}B)^{-1}CA^{-1},$$

to (23) gives

$$P(t) = P(t-1) - P(t-1)\hat{\phi}(t)[1 + \hat{\phi}^T(t)P(t-1)\hat{\phi}(t)]^{-1}\hat{\phi}^T(t)P(t-1). \quad (24)$$

Define the gain vector  $L(t) := P(t)\hat{\phi}(t) \in \mathbb{R}^{2n}$ . Post-multiplying (24) by  $\hat{\phi}(t)$ , we have

$$L(t) = P(t-1)\hat{\phi}(t)[1 + \hat{\phi}^T(t)P(t-1)\hat{\phi}(t)]^{-1}. \quad (25)$$

Thus, we have

$$P(t) = [I_{2n} - L(t)\hat{\phi}^T(t)]P(t-1). \quad (26)$$

Combining (22), (25), (26) and (21), we can summarize the estimated states based recursive least squares (ES-RLS) algorithm as [1]

$$\hat{\theta}(t) = \hat{\theta}(t-1) + L(t)[y(t) - \hat{\phi}^T(t)\hat{\theta}(t-1)], \quad \hat{\theta}(0) = \mathbf{1}_{2n}/p_0, \quad (27)$$

$$L(t) = P(t-1)\hat{\phi}(t)[1 + \hat{\phi}^T(t)P(t-1)\hat{\phi}(t)]^{-1}, \quad (28)$$

$$P(t) = [I_{2n} - L(t)\hat{\phi}^T(t)]P(t-1), \quad P(0) = p_0 I_{2n}, \quad (29)$$

$$\hat{\phi}(t) = \begin{bmatrix} \hat{\phi}(t) \\ \psi(t) \end{bmatrix}, \quad (30)$$

$$\hat{\phi}(t) = \begin{bmatrix} -\hat{x}_1(t-1) \\ -\hat{x}_1(t-2) \\ \vdots \\ -\hat{x}_1(t-n) \end{bmatrix}, \quad \psi(t) = \begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(t-n) \end{bmatrix}, \quad (31)$$

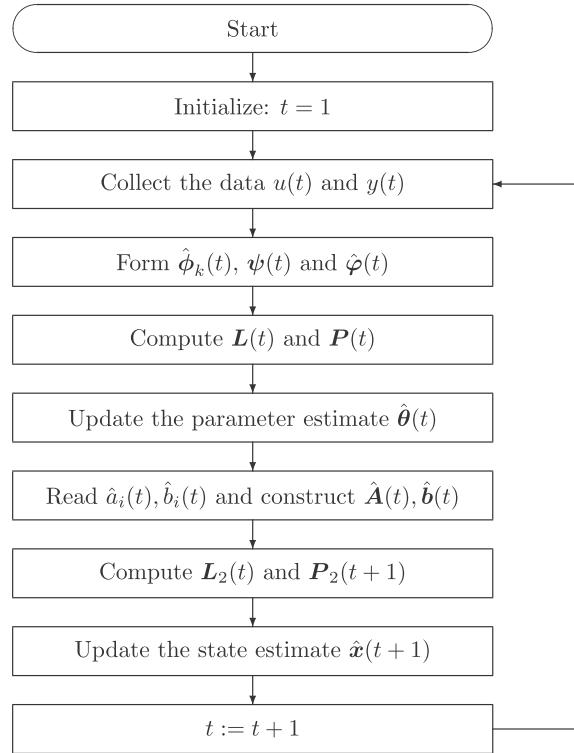
$$\hat{\theta}(t) = [\hat{a}_1(t), \hat{a}_2(t), \dots, \hat{a}_n(t), \hat{b}_1(t), \hat{b}_2(t), \dots, \hat{b}_n(t)]^T. \quad (32)$$

The algorithm in (27)–(32) compute recursively the parameter estimation vector  $\hat{\theta}(t)$  using the estimated states  $\hat{x}_1(t-i)$  in the information vector  $\hat{\phi}(t)$ .

Eqs. (27)–(32) form the estimated states based recursive least squares parameter identification algorithm for state space systems.

The following lists the steps of computing the parameter and state estimates for the algorithm in (27)–(32) and (15)–(18) with the data length  $k$  increasing.

1. To initialize, let  $t = 1, \hat{\theta}(0) = \mathbf{1}_{2n}/p_0, P(0) = p_0 I_{2n}, \hat{x}(t-i) = 1/p_0$  for  $i = 1, 2, \dots, n, P_2(1) = I_n, p_0 = 10^6$ .
2. Collect the input-output data  $u(t)$  and  $y(t)$ .



**Fig. 1.** The flowchart of computing the parameter estimate  $\hat{\theta}(t)$  and the state estimate  $\hat{x}(t)$ .

3. Form  $\hat{\phi}_k(t)$  and  $\psi(t)$  using (31) and  $\hat{\varphi}(t)$  by (30).
4. Compute the gain vector  $\mathbf{L}(t)$  and the covariance matrix  $\mathbf{P}(t)$  using (28) and (29), and update the parameter estimate  $\hat{\theta}(t)$  using (27).
5. Read  $\hat{a}_i(t)$  and  $\hat{b}_i(t)$  from  $\hat{\theta}(t)$  according to (32), and construct  $\hat{\mathbf{A}}(t)$  and  $\hat{\mathbf{b}}(t)$  using (18).
6. Compute the state gain vector  $\mathbf{L}_2(t)$  and the covariance matrix  $\mathbf{P}_2(t + 1)$  using (16) and (17), and update the state estimate  $\hat{x}(t + 1)$  using (15).
7. Increase  $t$  by 1 and go to step 2.

The flowchart of computing the parameter estimate  $\hat{\theta}(t)$  and the state estimate  $\hat{x}(t)$  is shown in Fig. 1.

#### 4. Example

Consider the following state space system:

$$\begin{aligned} \mathbf{x}(t+1) &= \begin{bmatrix} 0.8 & 1 \\ -0.4 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1.68 \\ 2.32 \end{bmatrix} u(t), \\ y(t) &= [1, 0] \mathbf{x}(t) + v(t). \end{aligned}$$

The parameter vector to be estimated is

$$\theta = [a_1, a_2, b_1, b_2]^T = [-0.80, 0.40, 1.68, 2.32]^T.$$

In simulation, the input  $\{u(t)\}$  is taken as an independent persistent excitation signal sequence with zero mean and unit variance, and  $\{v(t)\}$  as a white noise sequence with zero mean and variance  $\sigma^2 = 1.00^2$  and  $\sigma^2 = 2.00^2$ , respectively, the corresponding noise-to-signal ratios are  $\delta_{ns} = 21.14\%$  and  $\delta_{ns} = 42.28\%$ . Applying the combined parameter and state estimation algorithm in (27)–(32) and (15)–(18) to identify the parameters of this system. The parameter estimates and their estimation errors are shown in Tables 1 and 2, the parameter estimates  $\hat{a}_i(t)$  and  $\hat{b}_i(t)$  versus  $t$  are shown in Fig. 2 and the parameter estimation errors  $\delta$  versus  $t$  are shown in Fig. 3.

From Tables 1 and 2 and Figs. 2 and 3, we can see that the parameter estimation errors become smaller with the increasing of the data length  $t$ . This shows that the proposed algorithm works well.

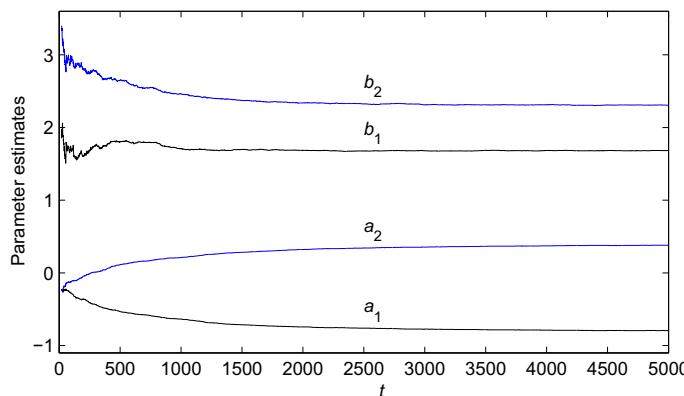
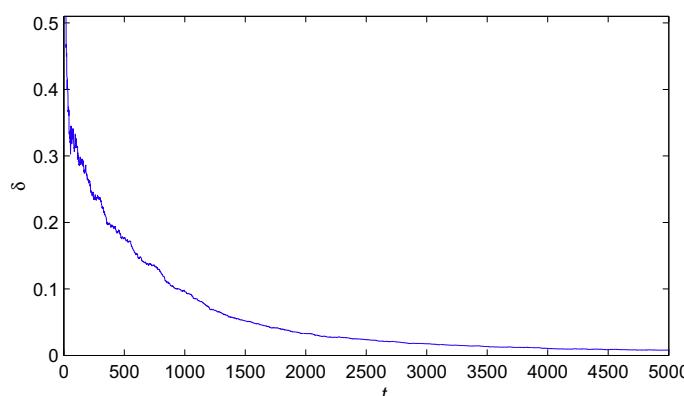
The following is the Matlab program of this example.

**Table 1**The parameter estimates and errors ( $\sigma^2 = 1.00^2$ ,  $\delta_{ns} = 21.14\%$ ).

$t$	$a_1$	$a_2$	$b_1$	$b_2$	$\delta$ (%)
100	-0.28240	-0.11921	1.68131	2.90353	31.22563
200	-0.35731	-0.04641	1.61350	2.80427	26.53828
500	-0.53120	0.11304	1.80451	2.65430	17.69214
1000	-0.63648	0.21173	1.71096	2.46077	9.59839
2000	-0.74319	0.32148	1.68539	2.33859	3.29339
3000	-0.77411	0.35447	1.68038	2.31114	1.77022
4000	-0.78798	0.37181	1.68284	2.31393	1.04523
5000	-0.79382	0.38083	1.68370	2.30665	0.81465
True values	-0.80000	0.40000	1.68000	2.32000	

**Table 2**The parameter estimates and errors ( $\sigma^2 = 2.00^2$ ,  $\delta_{ns} = 42.28\%$ ).

$t$	$a_1$	$a_2$	$b_1$	$b_2$	$\delta$ (%)
100	-0.27617	-0.10903	1.73217	3.08394	35.26457
200	-0.33908	-0.03543	1.66731	2.96745	30.20244
500	-0.50656	0.10603	1.86060	2.76488	21.15664
1000	-0.59426	0.18176	1.69254	2.52856	12.18110
2000	-0.71571	0.30026	1.66639	2.38384	4.86514
3000	-0.75423	0.33469	1.65991	2.34964	2.91328
4000	-0.77274	0.35482	1.66832	2.35992	2.23903
5000	-0.78088	0.36591	1.67271	2.34540	1.57235
True values	-0.80000	0.40000	1.68000	2.32000	

**Fig. 2.** The parameter estimation errors  $\delta$  versus  $t$  ( $\sigma^2 = 1.00^2$ ).**Fig. 3.** The parameter estimation error  $\delta$  versus  $t$  ( $\sigma^2 = 1.00^2$ ).

## 5. Conclusions

This paper proposes a combined parameter and state estimation algorithm for estimating the parameters and states of an observer canonical state space system. The simulation results indicate that the proposed algorithms are effective.

The proposed method can combine other methods, e.g., the multi-innovation identification methods [39–42], the hierarchical identification methods [43–45], the iterative identification methods [46], the two-stage identification algorithms and so on, to study identification problems of the controller canonical form, the controllability canonical form and the observability canonical form of scalar or multivariable systems [47–49].

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