Design and Analysis of a Novel $\mathcal{L}_1$ Adaptive Control Architecture With Guaranteed Transient Performance

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Abstract—**This paper presents a novel adaptive control architecture that adapts fast and ensures uniformly bounded transient response for system's both signals, input and output, simultaneously. This new architecture has a low-pass filter in the feedback loop and relies on the small-gain theorem for the proof of asymptotic stability. The tools from this paper can be used to develop a theoretically justified verification and validation framework for adaptive systems. Simulations illustrate the theoretical findings.**

Index Terms—**Fast and robust adaptation, guarteed transient performance, scaled response.**

I. INTRODUCTION

This paper presents a novel adaptive control architecture that leads to quantifiable performance bounds for a system's both signals, input and output, simultaneously. Performance bounds of adaptive controllers have been addressed in numerous publications [1]–[8], to name a few. However, as compared to linear systems theory, several important aspects of the transient performance analysis seem to be missing in these papers. First, all the bounds in these papers are computed for tracking errors only, and not for control signals. Although the latter can be deduced from the former, it is straightforward to verify that the ability to adjust the former may not extend to the latter in case of nonlinear control laws. Second, since the purpose of adaptive control is to ensure stable performance in the presence of modeling uncertainties, one needs to ensure that both signals of the system, input and output, retain uniform performance despite the changes in reference input and unknown parameters due to possible faults or unexpected disturbances. Finally, one needs to ensure that whatever modifications or solutions are suggested for performance improvement of adaptive controllers, they are not achieved via high-gain feedback.

In this paper, we define a new type of model following adaptive controller that adapts fast leading to desired transient performance for system's both input and output signals simultaneously. The small-gain theorem is invoked for the proof of asymptotic stability. The ideal (nonadaptive) version of this $\mathcal{L}_1$ adaptive controller is used along with the main system dynamics to define a closed-loop reference system, which gives an opportunity to estimate performance bounds in terms of $\mathcal{L}_\infty$ norms for the system's both signals. Design guidelines for the low-pass filter ensure that the closed-loop reference system approximates the desired system response, despite the fact that it depends upon the unknown parameter.

The paper is organized as follows. Section II states some preliminary definitions, and Section III gives the problem formulation. In Section IV, the new $\mathcal{L}_1$ adaptive controller is presented, the performance analysis of which is in Section V. Design guidelines are provided in Section VI. Unless otherwise mentioned, the notation $\| \cdot \|$ is

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used for the 2-norm of vectors, and $\chi(s)$ is used to denote the Laplace transform of $\chi(t)$.

II. PRELIMINARIES

In this section, we recall basic definitions and facts from linear systems theory [9], [10].

Definition 1: For a signal $\xi(t) = [\xi_1(t) \cdots \xi_n(t)]^\top \in \mathbb{R}^n$ defined for all $t \geq 0$, the truncated $\mathcal{L}_\infty$ norm and the $\mathcal{L}_\infty$ norm are $\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,\ldots,n} (\sup_{0 \leq \tau \leq t} |\xi_i(\tau)|)$, $\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,\ldots,n} (\sup_{\tau \geq 0} |\xi_i(\tau)|)$.

Definition 2: The $\mathcal{L}_1$ gain of an asymptotically stable and proper single-input single-output (SISO) system is defined as $\|H(s)\|_{\mathcal{L}_1} = \int_0^\infty |h(t)| dt$, where $h(t)$ is the impulse response of $H(s)$.

Definition 3: For an asymptotically stable and proper m input n output system $H(s)$, the $\mathcal{L}_1$ gain is defined as $\|H(s)\|_{\mathcal{L}_1} = \max_{i=1,\ldots,n} (\sum_{j=1}^m \|H_{ij}(s)\|_{\mathcal{L}_1})$, where $H_{ij}(s)$ is the ith row jth column entry of $H(s)$.

Lemma 1: For an asymptotically stable proper multi-input multi-output (MIMO) system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have

$$\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r_t\|_{\mathcal{L}_\infty} \qquad \forall t \geq 0. \qquad (1)$$

Corollary 1: For an asymptotically stable proper MIMO system $H(s)$, if the input $r(t) \in \mathbb{R}^m$ is bounded, then the output $x(t) \in \mathbb{R}^n$ is also bounded, and $\|x\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty}$.

Lemma 2: For a cascaded system $H(s) = H_2(s) H_1(s)$, where $H_1(s)$ and $H_2(s)$ are asymptotically stable proper systems, we have $\|H(s)\|_{\mathcal{L}_1} \leq \|H_2(s)\|_{\mathcal{L}_1} \|H_1(s)\|_{\mathcal{L}_1}$.

Theorem 1: ([9], Theorem 5.6) ($\mathcal{L}_1$ Small-Gain Theorem): The interconnected system $w_2(s) = \Delta(s)(w_1(s) - M(s)w_2(s))$ with input $w_1(t)$ and output $w_2(t)$ is asymptotically stable if $\|M(s)\|_{\mathcal{L}_1} \|\Delta(s)\|_{\mathcal{L}_1} < 1$.

Consider a linear time-invariant (LTI) system: $x(s) = (s\mathbb{I} - A)^{-1} b u(s)$ with Hurwitz $A \in \mathbb{R}^{n \times n}$ matrix, and let $(s\mathbb{I} - A)^{-1} b = n(s)/d(s)$, where $d(s) = \det(s\mathbb{I} - A)$, and $n(s)$ is an n-dimensional vector with its ith element being a polynomial function $n_i(s) = \sum_{j=1}^n n_{ij} s^{j-1}$.

Lemma 3: If $(A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n)$ is controllable, the matrix N with entries n_{ij} is full rank.

Proof: Controllability of (A, b) implies reachability. Hence, given an initial condition $x(t_0) = 0$ and arbitrary $x_{t_1} = x(t_1)$, there exists $u(\tau), \tau \in [t_0, t_1]$ such that $x(t_1) = x_{t_1}$. If N is not full rank, then there exists a $\mu \neq 0$, such that $\mu^\top n(s) = 0$. Thus, for $x(t_0) = 0$, one has $\mu^\top x(\tau) = 0, \forall \tau > t_0$. This contradicts $x(t_1) = x_{t_1}$, in which x_{t_1} was an arbitrary point. Thus, N must be full rank. $\square$

Lemma 4: If (A, b) is controllable and $(s\mathbb{I} - A)^{-1} b$ is asymptotically stable, there exists $c \in \mathbb{R}^n$ such that $c^\top (s\mathbb{I} - A)^{-1} b$ is minimum phase with relative degree 1.

Proof: Since $c^\top (s\mathbb{I} - A)^{-1} b = (c^\top N [s^{n-1} \cdots 1]^\top)/d(s)$, we choose $\bar{c} \in \mathbb{R}^n$ such that $\bar{c}^\top [s^{n-1} \cdots 1]^\top$ is an asymptotically stable $n-1$ order polynomial. Let $c = (N^{-1})^\top \bar{c}$. Then, $c^\top (s\mathbb{I} - A)^{-1} b = \bar{c}^\top [s^{n-1} \cdots 1]^\top / d(s)$ has relative degree 1 with all its zeros in the left half plane. $\square$

III. PROBLEM FORMULATION

Consider the following SISO system dynamics

$$\dot{x}(t) = Ax(t) + b(u(t) - \theta^\top x(t)) \qquad y(t) = c^\top x(t) \qquad x(0) = x_0 \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector (measurable), $u(t) \in \mathbb{R}$ is the control signal, $b, c \in \mathbb{R}^n$ are known constant vectors, A is a known

$n \times n$ matrix, (A, b) is controllable, the unknown parameter $\theta \in \mathbb{R}^n$ belongs to a given compact convex set $\theta \in \Omega$, and $y(t) \in \mathbb{R}$ is the regulated output.

The control objective is to design an adaptive controller to ensure that the system output $y(t)$ follows a given reference signal $r(t)$ with quantifiable transient and steady-state performance bounds.

IV. $\mathcal{L}_1$ ADAPTIVE CONTROLLER

Consider the following control structure

$$u(t) = u_1(t) + u_2(t) \qquad u_1(t) = -K^\top x(t) \qquad (3)$$

where K renders $A_m = A - bK^\top$ Hurwitz, while $u_2(t)$ is generated by the adaptive controller. It leads to the following system:

$$\dot{x}(t) = A_m x(t) - b\theta^\top x(t) + bu_2(t) \qquad y(t) = c^\top x(t) \qquad x(0) = x_0. \qquad (4)$$

For the linearly parameterized system in (4), we consider the following state predictor

$$\dot{\hat{x}}(t) = A_m \hat{x}(t) - b\hat{\theta}^\top(t)x(t) + bu_2(t) \qquad \hat{y}(t) = c^\top \hat{x}(t) \qquad \hat{x}(0) = x_0 \qquad (5)$$

along with the projection-type adaptive law for $\hat{\theta}(t)$

$$\dot{\hat{\theta}}(t) = \Gamma \mathrm{Proj}(\hat{\theta}(t), x(t)\tilde{x}^\top(t)Pb) \qquad \hat{\theta}(0) = \hat{\theta}_0 \qquad \tilde{x}(t) = \hat{x}(t) - x(t) \qquad (6)$$

where $\Gamma > 0$ is the adaptation gain, and $P = P^\top > 0$ solves $A_m^\top P + PA_m = -Q$ for some $Q > 0$. Let

$$u_2(s) = C(s)(\bar{r}(s) + k_g r(s)) \qquad k_g = 1/(c^\top H_o(0))$$
$$H_o(s) = (s\mathbb{I} - A_m)^{-1}b \qquad (7)$$

where $\bar{r}(t) = \hat{\theta}^\top(t)x(t)$, while $C(s)$ is an asymptotically stable and strictly proper transfer function with dc gain $C(0) = 1$. The $\mathcal{L}_1$ adaptive controller consists of (3), (5), (6), and (7), with K and $C(s)$ such that

$$\lambda \triangleq \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max} < 1 \qquad \bar{G}(s) = H_o(s)(C(s) - 1)$$

$$\theta_{\max} = \max_{\theta \in \Omega} \sum_{i=1}^n |\theta_i|. \qquad (8)$$

We notice that the condition in (8) can be straightforwardly satisfied by increasing the bandwidth of $C(s)$ (refer to Lemma 8 later in Section VI). We further notice that $\hat{x}(s) = \bar{G}(s)\bar{r}(s) + G(s)r(s) + (s\mathbb{I} - A_m)^{-1}x_0$, $G(s) = k_g H_o(s)C(s)$.

V. ANALYSIS OF $\mathcal{L}_1$ ADAPTIVE CONTROLLER

Consider the following *ideal* version of the adaptive controller in (3), (7):

$$u_{\mathrm{ref}}(s) = C(s)(\theta^\top x_{\mathrm{ref}}(s) + k_g r(s)) - K^\top x_{\mathrm{ref}}(s). \qquad (9)$$

Notice that $C(s) = 1$ leads to the reference model of model reference adaptive control (MRAC). The controller in (9) leads to the following relation

$$x_{\mathrm{ref}}(s) = H_o(s)(k_g C(s)r(s) + (C(s) - 1)\theta^\top x_{\mathrm{ref}}(s)) + (s\mathbb{I} - A_m)^{-1}x_0$$

$$y_{\mathrm{ref}}(s) = c^\top x_{\mathrm{ref}}(s) \qquad (10)$$

which can be explicitly solved for $x_{\mathrm{ref}}(s)$ yielding

$$x_{\mathrm{ref}}(s) = (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r(s) + x_{\mathrm{in}}(s)$$

$$x_{\mathrm{in}}(s) = (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}(s\mathbb{I} - A_m)^{-1}x_0. \qquad (11)$$

Lemma 5: If $\|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max} < 1$, then $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}$ and $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)$ are asymptotically stable.

Proof: It follows from Definition 3 that $\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} = \max_{i=1,\ldots,n}(\|\bar{G}_i(s)\|_{\mathcal{L}_1}(\sum_{j=1}^n |\theta_j|))$. We have $\sum_{j=1}^n |\theta_j| \le \theta_{\max}$, and hence, $\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} \le \max_{i=1,\ldots,n}(\|\bar{G}_i(s)\|_{\mathcal{L}_1})\theta_{\max} = \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max}$. The relationship in (8) implies that $\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} < 1$, and therefore, Theorem 1 ensures that the LTI system $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}$ is asymptotically stable. Lemma 2 implies that $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)$ is asymptotically stable. $\square$

Consider the Lyapunov function candidate: $V(\tilde{x}(t), \tilde{\theta}(t)) = \tilde{x}^\top(t)P\tilde{x}(t) + \tilde{\theta}^\top(t)\Gamma^{-1}\tilde{\theta}(t)$, where $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$. It follows from (4) and (5) that

$$\dot{\tilde{x}}(t) = A_m \tilde{x}(t) - b\tilde{\theta}^\top(t)x(t) \qquad \tilde{x}(0) = 0. \qquad (12)$$

It is straightforward to verify that $\dot{V}(t) \le -\tilde{x}^\top(t)Q\tilde{x}(t) \le 0$, which is independent of $u_2(t)$. This implies that $\tilde{x}(t)$ and $\tilde{\theta}(t)$ are bounded. To prove asymptotic convergence of $\tilde{x}(t)$ to zero, one needs to ensure that $\hat{x}(t)$ in (5) is uniformly bounded.

Lemma 6: For the system in (2) and the controller defined via (3), (5), (6), (7), and (8), we have:

$$\|\tilde{x}(t)\| \le \sqrt{\bar{\theta}_{\max}/(\lambda_{\min}(P)\Gamma)}, \ \forall t \ge 0$$

$$\bar{\theta}_{\max} \triangleq \max_{\theta \in \Omega} \sum_{i=1}^n 4\theta_i^2, \quad \text{and} \quad \lim_{t\to\infty}\tilde{x}(t) = 0. \qquad (13)$$

Proof: Since $\tilde{x}(0) = 0$, then $\lambda_{\min}(P)\|\tilde{x}(t)\|^2 \le V(t) \le V(0) = \tilde{\theta}^\top(0)\Gamma^{-1}\tilde{\theta}(0)$, where $\lambda_{\min}(P)$ is the minimum eigenvalue of P. Thus, $\|\tilde{x}(t)\|^2 \le V(0)/\lambda_{\min}(P)$. Therefore, $\|\tilde{x}(t)\| \le \sqrt{\bar{\theta}_{\max}/\lambda_{\min}(P)\Gamma}$, and also, $\|\tilde{x}_t\|_{\mathcal{L}_\infty} \le \sqrt{V(0)/\lambda_{\min}(P)}$. Notice that $\|\|\hat{x}_t\|_{\mathcal{L}_\infty} - \|x_t\|_{\mathcal{L}_\infty}\| \le \sqrt{V(0)/\lambda_{\min}(P)}$. The projection in (6) ensures $\hat{\theta}(t) \in \Omega$. Since $\|\bar{r}_t\|_{\mathcal{L}_\infty} \le \theta_{\max}\|x_t\|_{\mathcal{L}_\infty}$, substituting for $\|x_t\|_{\mathcal{L}_\infty}$ leads to $\|\bar{r}_t\|_{\mathcal{L}_\infty} \le \theta_{\max}(\|\hat{x}_t\|_{\mathcal{L}_\infty} + \sqrt{V(0)/\lambda_{\min}(P)})$. Lemma 1 implies $\|\hat{x}_t\|_{\mathcal{L}_\infty} \le \|\bar{G}(s)\|_{\mathcal{L}_1}\|\bar{r}_t\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty}$, which leads to $\|\hat{x}_t\|_{\mathcal{L}_\infty} \le (\lambda\sqrt{V(0)/\lambda_{\min}(P)} + \|G(s)\|_{\mathcal{L}_1}\|r_t\|_{\mathcal{L}_\infty})/(1 - \lambda)$. As a result, $\|\hat{x}_t\|_{\mathcal{L}_\infty}$ is finite for any $t \ge 0$, and hence, $\hat{x}(t)$ is bounded. Thus, $\tilde{x}(t)$ is bounded, and Barbalat's lemma implies that $\lim_{t\to\infty}\tilde{x}(t) = 0$. $\square$

Letting $r_1(t) = \tilde{\theta}^\top(t)x(t)$, notice that $\bar{r}(t) = \theta^\top(\hat{x}(t) - \tilde{x}(t)) + r_1(t)$. Hence, the state predictor can be rewritten as $\hat{x}(s) = (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}(-\bar{G}(s)\theta^\top\tilde{x}(s) + \bar{G}(s)r_1(s) + G(s)r(s)) + x_{\mathrm{in}}(s)$. It follows from (12) that $\tilde{x}(s) = -H_o(s)r_1(s)$. Using (8), the predictor can be presented as $\hat{x}(s) = (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r(s) + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}(-\bar{G}(s)\theta^\top\tilde{x}(s) - (C(s) - 1)\tilde{x}(s)) + x_{\mathrm{in}}(s)$. Using $x_{\mathrm{ref}}(s)$ from (11) and recalling the definition of $\tilde{x}(s) = \hat{x}(s) - x(s)$, one arrives at

$$x(s) = x_{\mathrm{ref}}(s) - (\mathbb{I} + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}(\bar{G}(s)\theta^\top + (C(s) - 1)\mathbb{I}))\tilde{x}(s). \qquad (14)$$

The expressions in (3), (7), and (9) lead to the following expression of the control signal

$$u(s) = u_{\mathrm{ref}}(s) + C(s)r_1(s) + (C(s)\theta^\top - K^\top)(x(s) - x_{\mathrm{ref}}(s)). \qquad (15)$$

We note that $(A - bK^\top, b)$ is the state space realization of $H_o(s)$. Since (A, b) is controllable, then $(A - bK^\top, b)$ is also controllable.

Lemma 4 implies that there exists $c_o \in \mathbb{R}^n$ and asymptotically stable polynomials $N_d(s)$ and $N_n(s)$ such that $c_o^\top H_o(s) = N_n(s)/N_d(s)$, where $\deg(N_d(s)) - \deg(N_n(s)) = 1$.

Theorem 2: For the system in (2) and the controller in (3), (5), (6), (7), and (8), we have

$$\lim_{t\to\infty} \|x(t) - x_{\mathrm{ref}}(t)\| = 0 \qquad \lim_{t\to\infty} |u(t) - u_{\mathrm{ref}}(t)| = 0 \quad (16)$$

$$\|x - x_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \le \gamma_1/\sqrt{\Gamma} \qquad \|u - u_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \le \gamma_2/\sqrt{\Gamma} \quad (17)$$

where $\gamma_1 = \|H_2(s)\|_{\mathcal{L}_1} \sqrt{\bar{\theta}_{\max}/\lambda_{\min}(P)}$, $H_2(s) = \mathbb{I} + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}(\bar{G}(s)\theta^\top + (C(s) - 1)\mathbb{I})$, $\gamma_2 = \|C(s)[1/c_o^\top H_o(s)]c_o^\top\|_{\mathcal{L}_1}\sqrt{\bar{\theta}_{\max}/\lambda_{\min}(P)} + \|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1}\gamma_1$.

Proof: Let $r_2(t) = x_{\mathrm{ref}}(t) - x(t)$. It follows from (14) that $r_2(s) = (\mathbb{I} + (\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}(\bar{G}(s)\theta^\top + (C(s) - 1)\mathbb{I}))\tilde{x}(s)$. The signal $r_2(t)$ can be viewed as the response of the LTI system $H_2(s)$ to the bounded error signal $\tilde{x}(t)$. Lemma 5 implies that $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}$, $\bar{G}(s)$, $C(s)$ are asymptotically stable, and therefore, $H_2(s)$ is asymptotically stable. Hence, from (13), we have $\lim_{t\to\infty} r_2(t) = 0$. Let $r_3(s) = C(s)r_1(s) + (C(s)\theta^\top - K^\top)(x(s) - x_{\mathrm{ref}}(s))$. It follows from (15) that $r_3(t) = u(t) - u_{\mathrm{ref}}(t)$, while the relationships in (12) and (13) imply that $\lim_{t\to\infty} r_1(t) = 0$, and therefore, $\lim_{t\to\infty} r_3(t) = 0$.

Using Lemma 1, from (13) one can derive the following upper bound $\|r_2\|_{\mathcal{L}_\infty} \le \|H_2(s)\|_{\mathcal{L}_1}\sqrt{\bar{\theta}_{\max}/\lambda_{\min}(P)\Gamma}$, which leads to $\|x - x_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \le \gamma_1/\sqrt{\Gamma}$. From $\tilde{x}(s) = -H_o(s)r_1(s)$, we have $r_3(s) = C(s)[c_o^\top H_o(s)r_1(s)]/[c_o^\top H_o(s)] + (C(s)\theta^\top - K^\top)(x(s) - x_{\mathrm{ref}}(s)) = -C(s)[1/c_o^\top H_o(s)]c_o^\top \tilde{x}(s) + (C(s)\theta^\top - K^\top)(x(s) - x_{\mathrm{ref}}(s))$. Since $C(s)$ is asymptotically stable and strictly proper, the complete system $C(s)[1/c_o^\top H_o(s)]$ is proper and asymptotically stable, which implies that its $\mathcal{L}_1$ gain is finite. Hence, $\|r_3\|_{\mathcal{L}_\infty} \le \|C(s)[1/c_o^\top H_o(s)]c_o^\top\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty} + \|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1}\|x - x_{\mathrm{ref}}\|_{\mathcal{L}_\infty}$, which leads to the second upper bound in (17). $\square$

From (11), it follows that $y_{\mathrm{ref}}(s) = c^\top(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r(s) + c^\top x_{\mathrm{in}}(s)$. If $r(t)$ is constant, the final value theorem ensures $\lim_{t\to\infty} y_{\mathrm{ref}}(t) = c^\top H_o(0)C(0)k_g r = r$, and hence, (16) implies $\lim_{t\to\infty} y(t) = r$.

Remark 1: Theorem 2 implies that by increasing the adaptive gain, the time histories of $x(t)$ and $u(t)$ can be made as close as possible to $x_{\mathrm{ref}}(t)$ and $u_{\mathrm{ref}}(t)$ for all $t \ge 0$. This, in turn, reduces the control objective to selection of K and $C(s)$ to ensure that the reference LTI system has the desired response.

Remark 2: Notice that if we set $C(s) = 1$, which corresponds to MRAC, $\|C(s)[1/c_o^\top H_o(s)]c_o^\top\|_{\mathcal{L}_1}$ cannot be finite, since $H_o(s)$ is strictly proper. Therefore, $\gamma_2 \to \infty$, and hence, for the control signal of MRAC, one cannot conclude a uniform performance bound from (17).

VI. DESIGN OF THE $\mathcal{L}_1$ ADAPTIVE CONTROLLER

Notice that the closed-loop reference system in (9) and (11) depends upon the unknown parameter θ. Consider the following signals

$$x_{\mathrm{des}}(s) = G(s)r(s) + x_{\mathrm{in}}(s) = C(s)k_g H_o(s)r(s) + x_{\mathrm{in}}(s)$$
$$y_{\mathrm{des}}(s) = c^\top x_{\mathrm{des}}(s) \quad (18)$$

$$u_{\mathrm{des}}(s) = k_g C(s)(1 + C(s)\theta^\top H_o(s) - K^\top H_o(s))r(s). \quad (19)$$

Lemma 7: Subject to (8), the following upper bounds hold

$$\|y_{\mathrm{ref}} - y_{\mathrm{des}}\|_{\mathcal{L}_\infty} \le \frac{\lambda}{1 - \lambda}\|c^\top\|_{\mathcal{L}_1}\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}$$

$$\|y_{\mathrm{ref}} - y_{\mathrm{des}}\|_{\mathcal{L}_\infty} \le \frac{1}{1 - \lambda}\|c^\top\|_{\mathcal{L}_1}\|h_3\|_{\mathcal{L}_\infty} \quad (20)$$

$$\|u_{\mathrm{ref}} - u_{\mathrm{des}}\|_{\mathcal{L}_\infty} \le \frac{\lambda}{1 - \lambda}\|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1}\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} \quad (21)$$

$$\|u_{\mathrm{ref}} - u_{\mathrm{des}}\|_{\mathcal{L}_\infty} \le \frac{1}{1 - \lambda}\|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1}\|h_3\|_{\mathcal{L}_\infty} \quad (22)$$

where $h_3(t)$ is the inverse Laplace transform of $H_3(s) = (C(s) - 1)C(s)r(s)k_g H_o(s)\theta^\top H_o(s)$.

Proof: It follows from (11) that $y_{\mathrm{ref}}(s) = c^\top(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)r(s) + c^\top x_{\mathrm{in}}(s)$. Following Lemma 5, the condition in (8) ensures the stability of the reference LTI system. Since $(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}$ is asymptotically stable, then one can expand it into convergent series and further write

$$y_{\mathrm{ref}}(s) = c^\top\left(\mathbb{I} + \sum_{i=1}^\infty (\bar{G}(s)\theta^\top)^i\right)G(s)r(s) + c^\top x_{\mathrm{in}}(s)$$
$$= y_{\mathrm{des}}(s) + c^\top\left(\sum_{i=1}^\infty (\bar{G}(s)\theta^\top)^i\right)G(s)r(s). \quad (23)$$

Let $r_4(s) = c^\top(\sum_{i=1}^\infty (\bar{G}(s)\theta^\top)^i)G(s)r(s)$. Then, $r_4(t) = y_{\mathrm{ref}}(t) - y_{\mathrm{des}}(t)$. Lemma 5 implies that $\|\bar{G}(s)\theta^\top\|_{\mathcal{L}_1} \le \lambda$, and it follows from Lemma 2 that

$$\|r_4\|_{\mathcal{L}_\infty} \le \left(\sum_{i=1}^\infty \lambda^i\right)\|c^\top\|_{\mathcal{L}_1}\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} = \frac{\lambda}{1-\lambda}\|c^\top\|_{\mathcal{L}_1}\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}. \quad (24)$$

From (23), we have $y_{\mathrm{ref}}(s) = y_{\mathrm{des}}(s) + c^\top(\sum_{i=1}^\infty (\bar{G}(s)\theta^\top)^{i-1})\bar{G}(s)\theta^\top G(s)r(s)$, which along with (8) leads to $y_{\mathrm{ref}}(s) = y_{\mathrm{des}}(s) + c^\top(\sum_{i=1}^\infty (\bar{G}(s)\theta^\top)^{i-1})H_3(s)$. Lemma 1 immediately implies that $\|r_4\|_{\mathcal{L}_\infty} \le (\sum_{i=1}^\infty \lambda^{i-1})\|c^\top\|_{\mathcal{L}_1}\|h_3\|_{\mathcal{L}_\infty}$. Comparing $u_{\mathrm{des}}(s)$ in (19) to $u_{\mathrm{ref}}(s)$ in (9), it follows that $u_{\mathrm{des}}(s)$ can be written as $u_{\mathrm{des}}(s) = k_g C(s)r(s) + (C(s)\theta^\top - K^\top)x_{\mathrm{des}}(s)$, where $x_{\mathrm{des}}(s) = C(s)k_g H_o(s)r(s) + x_{\mathrm{in}}(s)$. Therefore, $u_{\mathrm{ref}}(s) - u_{\mathrm{des}}(s) = (C(s)\theta^\top - K^\top)(x_{\mathrm{ref}}(s) - x_{\mathrm{des}}(s))$. Hence, it follows from Lemma 1 that $\|u_{\mathrm{ref}} - u_{\mathrm{des}}\|_{\mathcal{L}_\infty} \le \|C(s)\theta^\top - K^\top\|_{\mathcal{L}_1}\|x_{\mathrm{ref}} - x_{\mathrm{des}}\|_{\mathcal{L}_\infty}$. Using the same steps as for $\|y_{\mathrm{ref}} - y_{\mathrm{des}}\|_{\mathcal{L}_\infty}$, we have $\|x_{\mathrm{ref}} - x_{\mathrm{des}}\|_{\mathcal{L}_\infty} \le \lambda/(1 - \lambda)\|G(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}$, $\|x_{\mathrm{ref}} - x_{\mathrm{des}}\|_{\mathcal{L}_\infty} \le 1/(1 - \lambda)\|h_3\|_{\mathcal{L}_\infty}$, which leads to (21) and (22). $\square$

Taking into consideration that $x_{\mathrm{in}}(t)$ is exponentially decaying, the control objective can be achieved via selection of K and $C(s)$, such that $C(s)c^\top H_o(s)$ defines the desired transient and steady-state performance and $C(s)$ minimizes λ or $\|h_3\|_{\mathcal{L}_\infty}$. We note that $C(s)c^\top H_o(s)$ does not depend on the unknown parameters. Considering the following conservative upper bound $\lambda \le \|H_o(s)\|_{\mathcal{L}_1}\|C(s) - 1\|_{\mathcal{L}_1}\theta_{\max}$, the aforementioned objectives can be achieved from two different perspectives: 1) fix $C(s)$ and minimize $\|H_o(s)\|_{\mathcal{L}_1}$ and 2) fix $H_o(s)$ and minimize one of the following $\|H_o(s)(C(s) - 1)\|_{\mathcal{L}_1}$, $\|(C(s) - 1)r(s)\|_{\mathcal{L}_1}$, or $\|C(s)(C(s) - 1)\|_{\mathcal{L}_1}$ via the choice of $C(s)$.

1) High-Gain Design: Let $C(s)$ define the desired transient and steady-state performance. Then minimization of $\|H_o(s)\|_{\mathcal{L}_1}$ can be achieved by choosing K sufficiently large. Since $C(s)$ is a strictly proper system containing the dominant poles of the closed-loop system in $k_g c^\top H_o(s)C(s)$ and $k_g c^\top H_o(0) = 1$, we have $k_g c^\top H_o(s)C(s) \approx C(s)$. However, minimization of $\|H_o(s)\|_{\mathcal{L}_1}$ via large K hurts robustness.

2) MRAC Without High-Gain Feedback: As in MRAC, choose K such that $k_g c^\top H_o(s)$ defines the desired reference system.

Lemma 8: Let $C(s) = \omega/(s + \omega)$. For any strictly proper asymptotically stable system $H_o(s)$, the following is true: $\lim_{\omega\to\infty}\|(C(s) - 1)H_o(s)\|_{\mathcal{L}_1} = 0$.

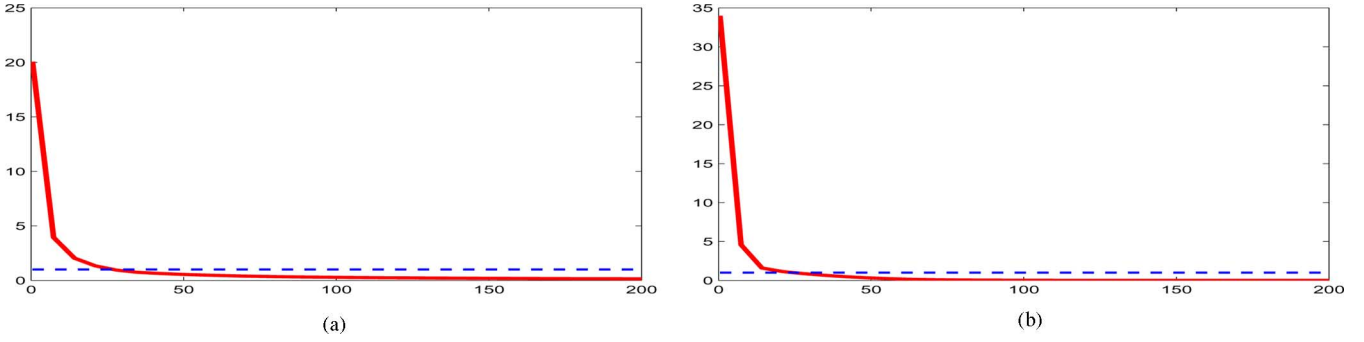Fig. 1. λ_1 and λ_2 (solid) with respect to ω and constant 1 (dashed).

Proof: Note that $(C(s) - 1)H_o(s) = -sH_o(s)/(s + \omega)$. Since $H_o(s)$ is strictly proper and asymptotically stable, $\|sH_o(s)\|_{\mathcal{L}_1}$ is finite, and hence, $\|(C(s) - 1)H_o(s)\|_{\mathcal{L}_1} \leq \|sH_o(s)\|_{\mathcal{L}_1}/\omega$. □

Thus, by increasing the bandwidth of $C(s)$, it is possible to render $\|\bar{G}(s)\|_{\mathcal{L}_1}$ arbitrarily small. With large ω, the pole $-\omega$ of $C(s)$ will be dominated by the poles of $H_o(s)$, implying that $k_g c^\top H_o(s)C(s) \approx k_g c^\top H_o(s)$.

However, increasing the bandwidth of $C(s)$ is not the only choice for minimizing $\|\bar{G}(s)\|_{\mathcal{L}_1}$. Since $C(s)$ is a low-pass filter, its complementary $1 - C(s)$ is a high-pass filter with its cutoff frequency approximating the bandwidth of $C(s)$. Then, $\bar{G}(s) = H_o(s)(C(s) - 1)$ is equivalent to cascading a low-pass system $H_o(s)$ with a high-pass system $C(s) - 1$. If one chooses the cutoff frequency of $C(s) - 1$ larger than the bandwidth of $H_o(s)$, the resulting $\bar{G}(s)$ is a "no-pass filter" with arbitrarily small $\mathcal{L}_1$ gain. This can be done via higher order filters.

To minimize $\|h_3\|_{\mathcal{L}_\infty}$, we note that $\|h_3\|_{\mathcal{L}_\infty}$ can be upperbounded in two ways: $\|h_3\|_{\mathcal{L}_\infty} \leq \|(C(s) - 1)r(s)\|_{\mathcal{L}_1}\|h_4\|_{\mathcal{L}_\infty}$, where $h_4(t)$ is the inverse Laplace transform of $H_4(s) = C(s)k_g H_o(s)\theta^\top H_o(s)$, and $\|h_3\|_{\mathcal{L}_\infty} \leq \|(C(s) - 1)C(s)\|_{\mathcal{L}_1}\|h_5\|_{\mathcal{L}_\infty}$, where $h_5(t)$ is the inverse Laplace transform of $H_5(s) = r(s)k_g H_o(s)\theta^\top H_o(s)$. Thus, $\|h_3\|_{\mathcal{L}_\infty}$ can be minimized by minimizing $\|(C(s) - 1)r(s)\|_{\mathcal{L}_1}$ or $\|(C(s) - 1)C(s)\|_{\mathcal{L}_1}$. Following the same arguments as before and assuming finite bandwidth for $r(t)$, one can choose the cutoff frequency of $C(s) - 1$ larger than the bandwidth of the reference signal $r(t)$ to minimize $\|(C(s) - 1)r(s)\|_{\mathcal{L}_1}$. Notice that if $C(s)$ is an ideal low-pass filter, then $C(s)(C(s) - 1) = 0$, and hence, $\|h_3\|_{\mathcal{L}_\infty} = 0$.

The earlier considerations ensure that $C(s) \approx 1$ in the bandwidth of $r(s)$ and $H_o(s)$. Since $k_g c^\top H_o(s)$ defines the desired performance, it follows from (18) that $C(s)k_g c^\top H_o(s) \approx k_g c^\top H_o(s)$.

Remark 3: Theorem 2 and Lemma 7 imply that the $\mathcal{L}_1$ adaptive controller can generate a system response to track (18) and (19) both in transient and steady state if we set the adaptive gain large and minimize λ or $\|h_3\|_{\mathcal{L}_\infty}$. Notice that $u_{\mathrm{des}}(t)$ in (19) depends upon the unknown parameter θ, while $y_{\mathrm{des}}(t)$ in (18) does not. This implies that for different values of θ, the $\mathcal{L}_1$ adaptive controller will generate different control signals (dependent on θ) to ensure uniform system response (independent of θ). This is natural, since different unknown parameters imply different systems, and to have similar response for different systems, the control signals have to be different. Here is the advantage of the $\mathcal{L}_1$ adaptive controller in a sense that it controls an unknown system as an LTI feedback controller would have done if the parameters were known.

Remark 4: It follows from Theorem 2 that in the presence of large adaptive gain, the $\mathcal{L}_1$ adaptive controller and the system state approximate $u_{\mathrm{ref}}(t), x_{\mathrm{ref}}(t)$. Therefore, $y(t)$ approximates the output response

of the LTI system $c^\top(\mathbb{I} - \bar{G}(s)\theta^\top)^{-1}G(s)$ to the input $r(t)$; hence, its transient performance specifications such as overshoot and settling time can be derived for every value of θ. If we further minimize λ or $\|h_3\|_{\mathcal{L}_\infty}$, it follows from Lemma 7 that $y(t)$ approximates the output response of the LTI system $C(s)c^\top H_o(s)$ to the input signal $r(t)$. In this case, the $\mathcal{L}_1$ adaptive controller leads to uniform transient performance of $y(t)$ independent of the value of the unknown parameter θ. For the resulting $\mathcal{L}_1$ adaptive control signal, one can characterize the transient specifications such as its amplitude and rate change for every $\theta \in \Omega$, using $u_{\mathrm{des}}(t)$.

Remark 5: We use a scalar system to compare the performance of the $\mathcal{L}_1$ adaptive controller and a linear high-gain controller. Let $\dot{x}(t) = -\theta x(t) + u(t)$, where $\theta \in [\theta_{\min}, \theta_{\max}]$. Let $u(t) = -kx(t) + kr(t)$, leading to $\dot{x}(t) = (-\theta - k)x(t) + kr(t)$. We need to choose $k > -\theta_{\min}$ to guarantee stability. We note that both the steady-state error and the transient performance depend on the unknown parameter value θ. By further introducing a proportional-integral controller, one can achieve zero steady-state error. If one chooses $k \gg \max\{|\theta_{\max}|, |\theta_{\min}|\}$, it leads to $x(s) = [k/(s - (-\theta - k))]r(s) \approx [k/(s + k)]r(s)$. To apply the $\mathcal{L}_1$ adaptive controller, let the desired reference system be $2/(s + 2)$. Let $u_1 = -2x, k_g = 2$, leading to $H_o(s) = 1/(s + 2)$. Choosing $C(s) = \omega_n/(s + \omega_n)$ with large ω_n, and setting the adaptive gain Γ large, it follows from (17) that $x(s) \approx x_{\mathrm{ref}}(s) \approx C(s)k_g H_o(s)r(s) \approx [\omega_n/(s + \omega_n)][2/(s + 2)]r(s) \approx [2/(s + 2)]r(s)$, $u(s) \approx u_{\mathrm{ref}}(s) = (-2 + \theta)x_{\mathrm{ref}}(s) + 2r(s)$. The first of these relationships implies that the control objective is met, while the second one states that $\mathcal{L}_1$ adaptive controller approximates $u_{\mathrm{ref}}(t)$, which cancels θ.

VII. SIMULATIONS

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}, b = [0 \quad 1]^\top, c = [1 \quad 0]^\top, \theta = [4 \quad -4.5]^\top$ in (2), and let $\Omega = \{\theta_1 \in [-10, 10], \theta_2 \in [-10, 10]\}$. Letting $K = 0$, $\Gamma = 10\,000$, we implement the controller following (3), (5), (6), and (7). Then, $\theta_{\max} = 20$, while $\|\bar{G}(s)\|_{\mathcal{L}_1}$ can be calculated numerically. In Fig. 1(a), we plot $\lambda_1 = \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max}$ with respect to ω, and notice that for $\omega > 30$, we have $\lambda_1 < 1$. Choosing $C(s) = 160/(s + 160)$ gives $\lambda_1 = \|\bar{G}(s)\|_{\mathcal{L}_1}\theta_{\max} = 0.1725 < 1$, which leads to improved performance bounds in (20)–(22). The simulation results of the $\mathcal{L}_1$ adaptive controller are shown in Fig. 2(a) and (b) for reference inputs $r = 25, 100, 400$. We note that it leads to scaled control inputs and scaled system outputs for scaled reference inputs. Fig. 3(a) and (b) shows the performance for $r(t) = 100 \cos(0.2t)$, without any retuning of the controller. We note that $\hat{\theta}^\top(t)x(t) - \theta^\top x(t)$ is a signal containing high-frequency harmonics and with zero dc component.
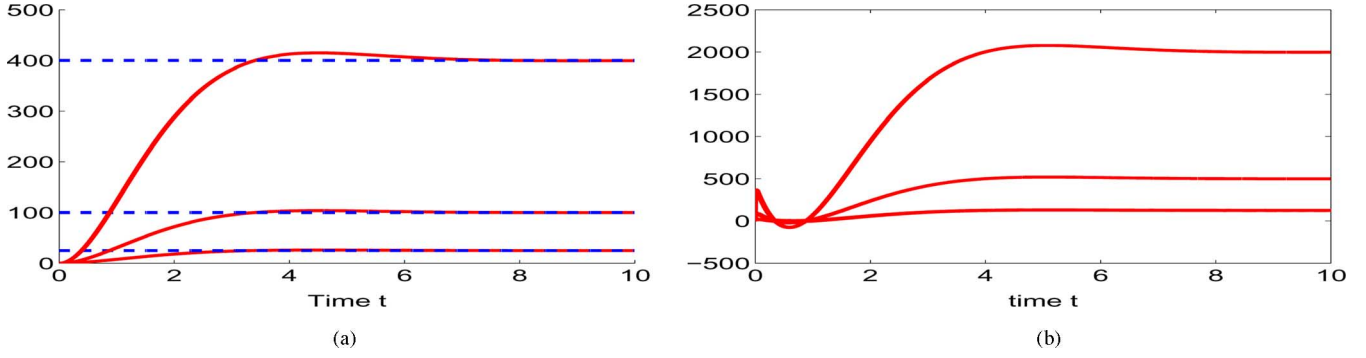
Fig. 2.   Performance of $\mathcal{L}_1$ adaptive controller with $C(s) = 160/(s + 160)$ for $r = 25, 100, 400$. (a) $y(t)$ (solid) and $r(t)$ (dashed). (b) Time-history of $u(t)$.
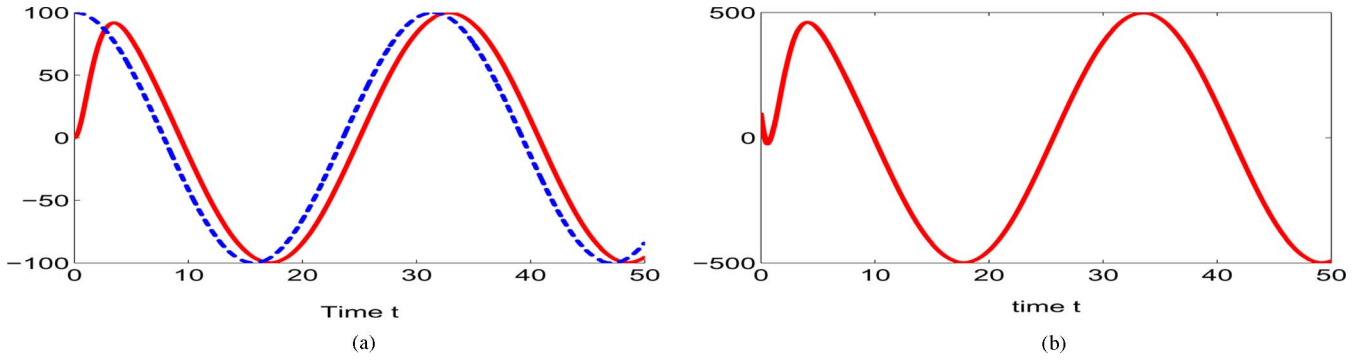

Fig. 3.   Performance of $\mathcal{L}_1$ adaptive controller with $C(s) = 160/(s + 160)$ for $r = 100 \cos (0.2t)$. (a) $y(t)$ (solid) and $r(t)$ (dashed). (b) Time-history of $u(t)$.
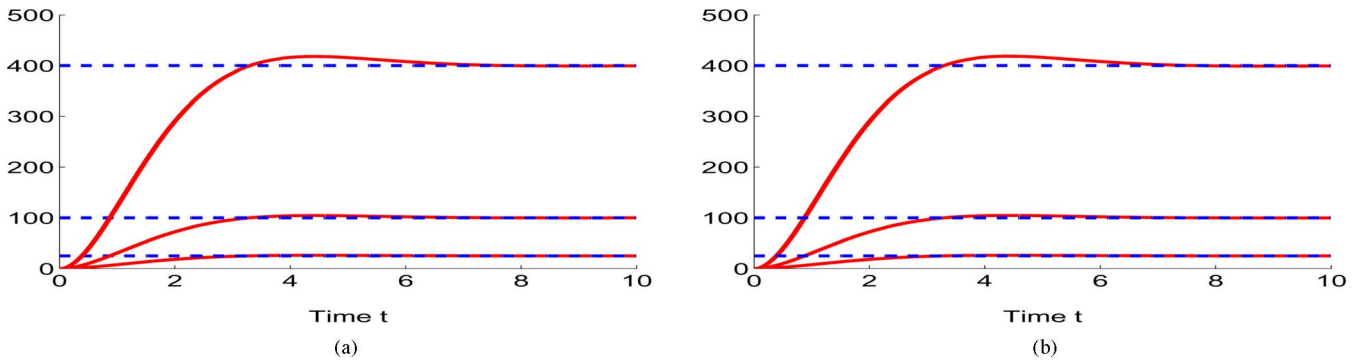

Fig. 4.   Performance of $\mathcal{L}_1$ adaptive controller with $C(s) = [7500\, s + 50^3]/(s + 50)^3$ for $r = 25, 100, 400$. (a) $y(t)$ (solid) and $r(t)$ (dashed). (b) Time-history of $u(t)$.

Next, let

$$\Gamma = 400 \qquad C(s) = \frac{3\omega^2 s + \omega^3}{(s + \omega)^3}. \qquad (25)$$

In Fig. 1(b), we plot $\lambda_2 = \|\bar{G}(s)\|_{\mathcal{L}_1} \theta_{\max}$ and notice that for $\omega > 25$, we have $\lambda_2 < 1$. Letting $\omega = 50$ leads to $\lambda_2 = 0.3984$. The simulation results are shown in Fig. 4(a) and (b) for reference inputs $r = 25, 100, 400$, which are again scaled for scaled reference inputs.

This example points out that with a higher order filter $C(s)$, one could use relatively small adaptive gain. While a rigorous relationship between the choice of the adaptive gain and the order of the filter cannot be derived, an insight into this can be gained from the following analysis. It follows from (2), (3), and (7) that $x(s) = G(s)r(s) -$

$H_o(s)\theta^\top x(s) + H_o(s)C(s)\bar{r}(s) + (s\mathbb{I} - A_m)^{-1}x_0$, while the state predictor can be rewritten as $\hat{x}(s) = G(s)r(s) + H_o(s)(C(s) - 1)\bar{r}(s) + (s\mathbb{I} - A_m)^{-1}x_0$. We note that the low-frequency component $C(s)\bar{r}(s)$ is the input to the system, while the complementary high-frequency component $(1 - C(s))\bar{r}(s)$ goes into the state predictor. If the bandwidth of $C(s)$ is large, then it can suppress only the high frequencies in $\bar{r}(t)$, which appear only in the presence of large adaptive gain. A properly designed higher order $C(s)$ can be more effective to serve the purpose of filtering with reduced tailing effects, and hence, can generate similar λ with smaller bandwidth. This further implies that similar performance can be achieved with smaller adaptive gain.

VIII. Conclusion

A novel adaptive control architecture is presented that leads to uniform transient response for a system's both signals simultaneously. Its performance bounds with respect to a reference LTI system imply that by increasing the adaptation gain, one can achieve scaled response for the system's both signals simultaneously. This consequently holds the promise for development of theoretically justified tools for the verification and validation of adaptive systems.

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Robust Feedback Control for a Class of Uncertain MIMO Nonlinear Systems

Jian Chen, A. Behal, and D. M. Dawson

Abstract—In this paper, a continuous feedback tracking controller is developed for a class of high-order multi-input multi-output (MIMO) nonlinear systems with an input gain matrix that has nonzero leading principal minors but can be nonsymmetric. Under the mild assumption that the signs of the leading minors of the control input gain matrix are known, the controller yields locally uniformly ultimately bounded (UUB) tracking while compensating for unstructured uncertainty in both the drift vector and the input matrix. First, a full-state feedback controller is designed based on limited assumptions on the structure of the system nonlinearities, and the singularity-free controller is proven to yield locally UUB tracking through a Lyapunov-based analysis. Then, it is shown that an output feedback control can be designed based on a high-gain observer. Simulation results are provided to illustrate the performance of the proposed control algorithm.

Index Terms—Lyapunov analysis, MIMO systems, nonlinear control, output feedback control, robust control.

I. Model Development

We consider a class of multi-input multi-output (MIMO) nonlinear systems having the form [16]

$$x^{(n)} = h(\mathbf{x}) + G(\mathbf{x})u \tag{1}$$

where $x^{(i)} \in \mathbb{R}^m$, $i = 0, 1, \ldots, n-1$ are the system states, $\mathbf{x} \triangleq [x^T \ \dot{x}^T \ \cdots \ (x^{(n-1)})^T]^T \in \mathbb{R}^{mn}$, $u(t) \in \mathbb{R}^m$ represents the control input, and $h(\mathbf{x}) \in \mathbb{R}^m$ and $G(\mathbf{x}) \in \mathbb{R}^{m \times m}$ are uncertain $\mathcal{C}^2$ nonlinearities. We assume that $G(\mathbf{x})$ is a real matrix with nonzero leading principal minors. Based on [4] and [11], the real matrix $G(\mathbf{x})$ can be decomposed as $G(\mathbf{x}) = S(\mathbf{x})DU(\mathbf{x})$ where $S(\mathbf{x}) \in \mathbb{R}^{m \times m}$ is symmetric positive definite, $U(\mathbf{x}) \in \mathbb{R}^{m \times m}$ is unity upper triangular, and $D \triangleq \text{diag}\{\text{sgn}(d_1), \text{sgn}(d_2), \ldots, \text{sgn}(d_m)\} \in \mathbb{R}^{m \times m}$ is a diagonal matrix with diagonal entries $+1$ or -1 where $d_1 \triangleq \triangle_1$, $d_i \triangleq \triangle_i / \triangle_{i-1}$ $\forall i = 2, 3, \ldots, m$, and $\triangle_1, \triangle_2, \ldots, \triangle_m$ are leading principal minors of $G(\mathbf{x})$. For control design purposes, we assume that D is known.

After time differentiating (1), the following expression can be obtained

$$x^{(n+1)} = \varphi(\mathbf{x}, x^{(n)}) + G(\mathbf{x})\dot{u} \tag{2}$$

where $\varphi(\mathbf{x}, x^{(n)}) \in \mathbb{R}^m$ is defined as follows:

$$\varphi(\mathbf{x}, x^{(n)}) \triangleq \dot{h}(\mathbf{x}) + \dot{G}(\mathbf{x})G^{-1}(\mathbf{x})(x^{(n)} - h(\mathbf{x})).$$

Invoking the matrix decomposition property, (2) can be rewritten as

$$M(\mathbf{x})x^{(n+1)} = f(\mathbf{x}, x^{(n)}) + DU(\mathbf{x})\dot{u} \tag{3}$$